MTH 264, Final Exam, Spring 2024 Name KEY

Instructions: Work the problems below as directed. Show all work. Clearly mark your final answers. Use exact values unless the problem specifically directs you to round. Simplify as much as possible. Partial credit is possible, but solutions without work will not receive full credit.

Part 1: These questions you will submit answers to in Canvas. Show all work and submit the work with Part 2 of the exam. But you must submit the answers in Canvas to receive credit. Each question/answer will be listed separately. The Canvas question will refer to the number/part to indicate where you should submit which answer. The questions will appear in order (in case there is an inadvertent typo). Correct answers will receive full credit with or without work in this section, but if you don't submit work and clearly label your answers, you won't be able to challenge any scoring decisions for making an error of any kind.

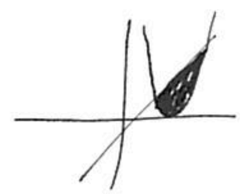
1. Find the area bounded by the curves $f(x) = \sqrt{\text{[redacted]}}$ 2x-3, and $g(x) = (\text{[redacted]})^2$ x-3 using the Fundamental Theorem of Calculus. Show all work. You may check your answers with your calculator. Report your answer in exact terms. (10 points)

$2x-3 = (x-3)^2$

$2x-3 = x^2-6x+9$

$0 = x^2-8x+12$

$(x-2)(x-6)$

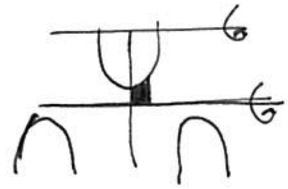


$$\int_2^6 2x-3-(x-3)^2\,dx = x^2-3x-\tfrac{1}{3}(x-3)^3\Big|_2^6$$

$$= 32/3$$

2. Find the volume of the solid of revolution generated by revolving the area bounded by the curves $y = \sec x, y = 0$ *and* $0 \le x \le \pi/3$ around a) the *x*-axis, and b) the line $y = 4$. Use the disk method (or washer method as appropriate) for both parts. [You should set up the equations and then you may integrate them in your calculator.] (10 points)

$$\pi\int_0^{\pi/3} (\sec x)^2\,dx = \pi\tan x\Big|_0^{\pi/3} = \pi\sqrt{3}$$



$$\pi\int_0^{\pi/3} (4)^2 - (4-\sec x)^2\,dx =$$

$$\pi\int_0^{\pi/3} 16 - (16 - 8\sec x + \sec^2 x)\,dx =$$

$$\pi\int_0^{\pi/3} 8\sec x - \sec^2 x\,dx =$$

$$\pi\left[8\ln|\sec x+\tan x| - \tan x\Big|_0^{\pi/3}\right] = \pi\left[8\ln|2+\sqrt{3}| - \sqrt{3}\right]$$

3. Find the length of arc of the curve $y = ln[sin(x)]$ over the interval $\left[\frac{\pi}{4}, \frac{3\pi}{4}\right]$. (8 points)

$$y' = \frac{1}{\sin x} \cdot \cos x = \cancel{\cot x}$$

$$\int_{\pi/4}^{3\pi/4} \sqrt{1+\cot^2 x}\ dx = \int_{\pi/4}^{3\pi/4} \csc x\ dx$$

$$= -\ln|\csc x + \cot x| \Big|_{\pi/4}^{3\pi/4} = -\ln|\sqrt{2}-1| + \ln|\sqrt{2}+1|$$

4. Find the area of the surface of revolution generated by the graph $y = \sqrt{9-x^2}$ over the interval [-3, 3] revolved around the x-axis. (10 points)

$$y' = \frac{1}{2}(9-x^2)^{-1/2} \cdot (-2x) = \frac{-x}{\sqrt{9-x^2}}$$

$$2\pi \int_{-3}^{3} \sqrt{9-x^2}\ \sqrt{1+\frac{x^2}{9-x^2}}\ dx =$$

$$2\pi \int_{-3}^{3} \sqrt{9-x^2}\ \sqrt{\frac{9-x^2+x^2}{9-x^2}}\ dx = 2\pi \int_{-3}^{3} \cancel{\sqrt{9-x^2}}\ \sqrt{\frac{9}{\cancel{9-x^2}}} = \int_{-3}^{3} 2\pi \cdot 3\ dx =$$

$$6\pi(6) = 36\pi$$

$$SA = 4\pi r^2 = 36\pi$$

5. A lunar module weighs 12 tons on the surface of the Earth. How much work is done in propelling the module from the surface of the moon to a height of 50 miles? Consider the radius of the moon to be 1100 miles and its force of gravity to be one sixth that of Earth. (10 points)

assume radius of the earth is about 4000 miles

$$F = \frac{k}{x^2}$$

$$12 \text{ tons} = \frac{k}{(4000)^2} \rightarrow k = 1.92 \times 10^8$$

$$\int_{4000}^{4050} 1.92\times10^8\ x^{-2}\ dx = -1.92\times10^8\ x^{-1} \Big|_{4000}^{4050} = 1.92\times10^8 \left(\frac{1}{4000} - \frac{1}{4050}\right)$$

$$\approx 592.59 \text{ ton-miles}$$

50 miles from surface of Earth

$$F = \frac{k}{x^2} \rightarrow 2 = \frac{k}{(1100)^2} = 2.42\times10^6$$

from moon

$$\int_{1100}^{1150} 2.42\times10^6\ x^{-2}\ dx = -2.42\times10^6\ x^{-1} \Big|_{1100}^{1150}\ 2.42\times10^6\left(\frac{1}{1100} - \frac{1}{1150}\right) \approx$$

95.65 ton-miles

6. For the integral $\int_1^3 \sqrt[3]{x}dx$, first calculate the number of subdivisions n that will be needed to have an Error using Simpson's Rule of less than or equal to 0.001, and then calculate the value of the integral using that method. (16 points)

$\Delta x = \frac{3-1}{4} = \frac{2}{4} = \frac{1}{2}$

$f(x) = x^{1/3}$
$f'(x) = \frac{1}{3}x^{-2/3}$
$f''(x) = -\frac{2}{9}x^{-5/3}$
$f'''(x) = \frac{10}{27}x^{-8/3}$
$f^{iv}(x) = \frac{-80}{81}x^{-11/3}$
max at $x=1$
$|80/81|$

$E \le \frac{|80/81|(3-1)^5}{180n^4}$

$n^4 \ge \frac{80/81\,(32)}{180}(1000)$

$= 175.58$

$\frac{1}{8}[\sqrt[3]{1} + 4\sqrt[3]{1.5} + 2\sqrt[3]{2} + 4\sqrt[3]{2.5} + \sqrt[3]{3}]$

$n \approx \sqrt[4]{175}$ $\approx 1.871222...$

$n = 4$

7. Find the slope of the tangent line to the graph $r = 1 + 2\cos\theta$ when $\theta = \frac{\pi}{6}$. You may use the formula $\frac{dy}{dx} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}$. (8 points)

$r' = -2\sin\theta$

$$\frac{dy}{dx} = \frac{-2\sin\theta(\sin\theta) + (1+2\cos\theta)\cos\theta}{-2\sin\theta\cos\theta - (1+2\cos\theta)\sin\theta} = \frac{-2\sin^2\theta + \cos\theta + 2\cos^2\theta}{-2\sin\theta\cos\theta - \sin\theta - 2\sin\theta\cos\theta}$$

$$= \frac{-2(1/2)^2 + \sqrt{3}/2 + 2\left(\frac{\sqrt{3}}{2}\right)^2}{-2(1/2)\frac{\sqrt{3}}{2} - \frac{1}{2} - 2(1/2)\frac{\sqrt{3}}{2}} = \frac{-1/2 + \frac{3}{2} + \frac{\sqrt{3}}{2}}{-\frac{\sqrt{3}}{2} - \frac{1}{2} - \frac{\sqrt{3}}{2}} = \frac{1 + \frac{\sqrt{3}}{2}}{-\sqrt{3} - 1/2} = \frac{2+\sqrt{3}}{-2\sqrt{3}-1}$$

8. Find the value that the series converges to (all of these converge to some known value). (7 points each)

a. $\sum_{k=1}^{\infty} \frac{2}{(2k+3)(2k+1)}$

$\frac{A}{2k+3} + \frac{B}{2k+1} \rightarrow A2k + A + 2Bk + 3B = 2$

$2A + 2B = 0 \rightarrow A + B = 0$

$A + 3B = 2 \rightarrow -B + 3B = 2 \rightarrow 2B = 2 \rightarrow B = 1$

$A = -1$

$\sum_{k=1}^{\infty}\left(\frac{1}{2k+1} - \frac{1}{2k+3}\right)$

$\lim_{k\to\infty} \frac{1}{2k+3} = 0$

$\frac{1}{3}$

Converges by telescoping Series test

b. $\sum_{k=0}^{\infty} 3^n 5^{-n} = \sum_{k=0}^{\infty}\left(\frac{3}{5}\right)^n$ $r < 1$ converges

Sum $= \frac{1}{1-3/5} = \frac{1}{2/5} = \frac{5}{2}$

by geometric series test

c. $\sum_{n=1}^{\infty} \frac{3^n}{n!} = e^3$

$$\lim_{n\to\infty} \left|\frac{3^{n+1}}{(n+1)!} \cdot \frac{n!}{3^n}\right| = \lim_{n\to\infty} \left|\frac{3}{n+1}\right| = 0$$

Converges by ratio test

9. Perform three steps of Euler's method on the differential equation $y' = e^{xy}$, starting at $(x, y) = (0,1)$. Use $\Delta x = h = 0.2$. What is your estimate for y_3? (10 points)

$m_1 = e^{0.1} = 1$

$y_1 = 1(0.2) + 1 = 1.2$

$m_2 = e^{0.2 \cdot 1.2} = e^{0.24}$

$y_2 = e^{0.24}(0.2) + 1.2 = 1.45424983$

$m_3 = e^{0.4 \cdot 1.4542983}$

$y_3 = e^{0.581699932}(0.2) + 1.4542983$

$= 1.812065261$

10. Determine whether the polar conics below are circles, ellipses, parabolas or hyperbolas. What is the eccentricity of each graph. (5 points each)

a. $r = \frac{5}{1-3\sin}$ $e = 3$ hyperola

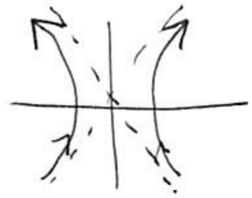
b. $r = 4\cos\theta - 1$ $e = 0$ circle

c. $r = \frac{1}{3+2\cos\theta}$ $e = \frac{2}{3}$ ellipse

d. $r = \frac{8}{1-\sin\theta}$ $e = 1$ parabola

11. Plot the set of parametric equations $x = 3\sec t$, $y = 7\tan t$. Clearly label its orientation. Rewrite the parametric equations as a single vector-valued function. (10 points)

$r(t) = \langle 3\sec t, 7\tan t \rangle$



12. Find the area of one petal of the graph $r = 4\cos 3\theta$. (10 points)

$4\cos 3\theta = 0$

$\cos 3\theta = 0$

$3\theta = \pi/2, 3\pi/2 \ldots$

$\theta = \pi/6, -\pi/6$

$2 \cdot \frac{1}{2} \int_0^{\pi/6} (4\cos 3\theta)^2 \, d\theta =$

$\int_0^{\pi/6} 16\cos^2 3\theta \, d\theta = 8\int_0^{\pi/6} 1 + \cos 6\theta \, d\theta$

$8\left[\theta + \frac{1}{6}\cancel{\sin 6\theta}\right]_0^{\pi/6}$

$8 \cdot \pi/6 = \frac{3\pi}{4}$

Part 2: In this section you will record your answers on paper along with your work. After scanning, submit them to a Canvas dropbox as directed. These questions will be graded by hand.

13. Integrate $\int x\sqrt{4-x}\,dx$ **both** by integration by parts, and by change of variables. Verify that your answers are algebraically equivalent (give or take a constant). (14 points)

$u = x \quad dv = (4-x)^{1/2}dx$

$du = dx \quad v = -2/3(4-x)^{3/2}$

$-\frac{2}{3}x(4-x)^{3/2} + \frac{2}{3}\int (4-x)^{3/2}dx =$

$-\frac{2}{3}x(4-x)^{3/2} - \frac{2}{3}\cdot\frac{2}{5}(4-x)^{5/2} + C$

$u = \sqrt{4-x}$

$u^2 = 4-x$

$x = 4-u^2$

$dx = -2u\,du$

$\int (4-u^2)u(-2u)\,du =$

$\int -8u^2 + 2u^4\,du$

$-\frac{8}{3}u^3 + \frac{2}{5}u^5 + C$

$-\frac{8}{3}(4-x)^{3/2} + \frac{2}{5}(x-4)^{5/2} + C$

14. Integrate using an appropriate method. (12 points each)

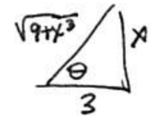
a. $\int \cos^4 x \sin^3 x\, dx$

$\int \cos^4 x (1 - \cos^2 x) \sin x\, dx$ $\quad$ $u = \cos x$ $\quad$ $du = -\sin x\, dx$

$-\int u^4 (1 - u^2)\, du = \int -u^4 + u^6\, du$

$\frac{1}{7}u^7 - \frac{1}{5}u^5 + C = \frac{1}{7}\cos^7 x - \frac{1}{5}\cos^5 x + C$

b. $\int \frac{\sqrt{9+x^2}}{x} dx$ $\quad$ $x = 3\tan\theta$ $\quad$ $dx = 3\sec^2\theta\, d\theta$ $\quad$ $\sqrt{9+x^2} = 3\sec\theta$



$\int \frac{3\sec\theta \cdot 3\sec^2\theta}{3\tan\theta} d\theta = \int \frac{9\sec^3\theta}{3} \cdot \frac{\cos\theta}{\sin\theta} d\theta = \int 3\sec^2\theta \cdot \frac{1}{\sin\theta} d\theta$

$3\int (1 + \tan^2\theta) \frac{1}{\sin\theta} d\theta = 3\int \csc\theta + \frac{\sin\theta}{\cos^2\theta} d\theta = 3\int \csc\theta + \sec\theta\tan\theta\, d\theta$

$3[-\ln|\csc\theta + \cot\theta| + \sec\theta] + C = -3\ln\left|\frac{\sqrt{9+x^2}}{x} + \frac{3}{x}\right| + 3\frac{\sqrt{9+x^2}}{3} + C$

c. $\int \frac{x^3-1}{x^2+x} dx = \frac{(x-1)(x^2+x+1)}{x(x+1)}$

$$\begin{array}{r} x - 1 \\ x^2 + x \overline{) \; x^3 \qquad -1} \\ -x^3 \mp x^2 \qquad \\ \hline -x^2 \quad -1 \\ +x^2 + x \\ \hline x - 1 \end{array}$$

$\int x - 1 + \frac{x-1}{x(x+1)} dx$ $\quad$ $\frac{A}{x} + \frac{B}{x+1} =$

$Ax + A + Bx = x - 1$

$A + B = 1$

$A = -1 \rightarrow B = 2$

$\int x - 1 - \frac{1}{x} + \frac{2}{x+1} dx = \frac{1}{2}x^2 - x - \ln x + 2\ln|x+1| + C$

15. Determine if the infinite series converge or diverge. Explain your reasoning, and which test you used to determine it. (6 points each)

a. $\sum_{k=2}^{\infty} \frac{ln^2(k)}{k}$ $\quad$ $\int u^2\, du$ $\quad$ $\frac{1}{3}(\ln x)^3 \Big|_2^{\infty} = \infty$

diverges by integral test

b. $\sum_{k=1}^{\infty} k e^{-k^2}$ $\quad$ $\int_1^{\infty} x e^{-x^2} dx = -\frac{1}{2}e^{-x^2}\Big|_1^{\infty} = 0 + \frac{1}{2}e^{-1}$

Converges, by integral test

c. $\sum_{k=1}^{\infty} \frac{k!}{k^k}$ $\quad$ $\lim_{k\to\infty} \left| \frac{(k+1)!}{(k+1)^{k+1}} \cdot \frac{k^k}{k!} \right| = \lim_{k\to\infty} \left| \frac{k^k}{(k+1)^k} \right| = e^{-1}$

Converges, by ratio test

d. $\sum_{k=1}^{\infty}(-1)^k k^{1/k}$

$$\lim_{k\to\infty} \sqrt[k]{k} = 1 \neq 0$$

diverges by alternating series test

16. Rewrite the expression $y = \frac{9x^2}{(x^3-1)^2}$ as a power series. (12 points)

$$= \frac{9x^2}{(1-x^3)^2}$$

$$\sum_{n=0} ar^n = a(1-r)^{-1}$$

$$\sum_{n=1} anr^{n-1} = \sum_{n=0} a(n+1)r^n = a(1-r)^{-2}$$

$$a = 9x^2, \quad r = x^3$$

$$\sum_{n=0}^{\infty} 9x^2(n+1)(x^3)^n = \sum_{n=0}^{\infty} 9(n+1)x^{3n+2}$$

17. To what value (if any) does the geometric series $\sum_{n=0}^{\infty}(-5)^n 2^{-n}$ converge? If it fails to converge, explain why. (6 points)

$$= \sum \left(\frac{-5}{2}\right)^n$$

diverges, by the geometric series test

$$\frac{5}{2} > 1$$

18. Find $\lim_{x\to 0} \frac{e^x + \ln(x+1) - 1}{x}$ using a power series. (8 points)

$$\lim_{x\to 0} \frac{\cancel{1} + x + x^2/2 + \dots + x + x^2/2 + \dots \cancel{-1}}{x}$$

$$\lim_{x\to 0} 1 + x/2 + \dots 1 + x/2 + \dots = 2$$

19. Find the solution to the initial value problem $y'(x) = \frac{\sqrt{y}}{3x}, y(1) = 4$. (6 points)

$$\frac{dy}{\sqrt{y}} = \frac{1}{3x}dx$$

$$y^{-1/2}dy$$

$$\int y^{-1/2}dy = \int \frac{1}{3}\cdot\frac{1}{x}dx$$

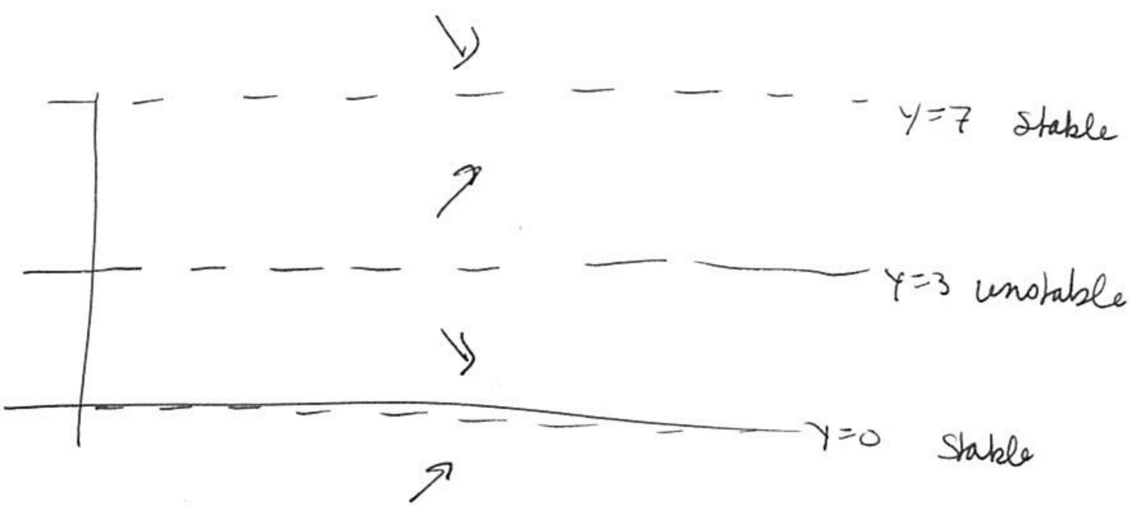
$$2y^{1/2} = \frac{1}{3}\ln x + C$$

$$2(\sqrt{4}) = \cancel{\frac{1}{3}\ln(1)}^{0} + C$$

$$4 = C$$

$$2\sqrt{y} = \frac{1}{3}\ln x + C$$

20. Given the differential equation $\frac{dy}{dx} = y(7-y)(y-3)$. Sketch the phase **plane** for the equation and use that information to graph the key features of the direction field such as the equilibria (steady state solutions) and the sign of the slope in each region. Label each equilibrium as stable, unstable or semi-stable. (12 points)



21. Solve the separable differential equation $(1-x^3)y' - 3x^2y = 0$. (10 points)

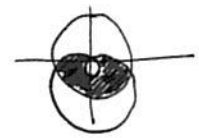
$$y' = \frac{3x^2 y}{(1-x^3)}$$

$$\int \frac{dy}{y} = \int \frac{3x^2}{(1-x^3)}dx$$

$$\ln y = -\ln|1-x^3| + C$$

$$y = A\ln\left|\frac{1}{1-x^3}\right|$$

22. Find the area of the region common to the circle $r = 2$, and $r = 1 - 2\sin\theta$. Sketch the region. (12 points)



$2 = 1 - 2\sin\theta$

$1 = -2\sin\theta$

$-\frac{1}{2} = \sin\theta$

$\theta = 7\pi/6,\ 11\pi/6$

$$\frac{1}{2}\int_{7\pi/6}^{11\pi/6} (2^2)\, d\theta + \frac{1}{2}\int_{-\pi/6}^{7\pi/6} (1-2\sin\theta)^2\, d\theta$$

$$2\left(\pi\cdot\frac{11}{6} - \pi\cdot\frac{7}{6}\right) + \frac{1}{2}\int_{-\pi/6}^{7\pi/6} 1 - 4\sin\theta + 4\sin^2\theta\, d\theta$$

$$2\left(\frac{2}{3}\pi\right) + \frac{1}{2}\int_{-\pi/6}^{7\pi/6} 1 - 4\sin\theta + 2 + 2\cos 2\theta\, d\theta$$

$$\frac{4}{3}\pi + \frac{1}{2}\int_{-\pi/6}^{7\pi/6} 3 - 4\sin\theta + 2\cos 2\theta\, d\theta =$$

$$\frac{4}{3}\pi + \frac{1}{2}\left[3\theta - 4\cos\theta + \sin 2\theta \right|_{-\pi/6}^{7\pi/6} =$$

$$\frac{4}{3}\pi + \frac{1}{2}\left[3(7\pi/6 - (-\pi/6)) + \frac{4\sqrt{3}}{2} + \frac{4\sqrt{3}}{2} + \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2}\right]$$

$8\pi/6$

$4\pi \quad + 4\sqrt{3} \quad + \sqrt{3}$

$$= \frac{4}{3}\pi + 2\pi + \frac{5\sqrt{3}}{2}$$

$$\frac{10\pi}{3} + \frac{5\sqrt{3}}{2}$$