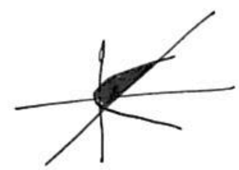
MTH 264, Exam #1, Spring 2024 Name KEY

Instructions: Work the problems below as directed. Show all work. Clearly mark your final answers. Use exact values unless the problem specifically directs you to round. Simplify as much as possible. Partial credit is possible, but solutions without work will not receive full credit.

Part 1: These questions you will submit answers to in Canvas. Show all work and submit the work with Part 2 of the exam. But you must submit the answers in Canvas to receive credit. Each question/answer will be listed separately. The Canvas question will refer to the number/part to indicate where you should submit which answer. The questions will appear in order (in case there is an inadvertent typo). Correct answers will receive full credit with or without work in this section, but if you don't submit work and clearly label your answers, you won't be able to challenge any scoring decisions for making an error of any kind.

1. Find the area of the region bounded by the curves $f(y) = y^2$, and $g(y) = y + 2$. Sketch the graph. (10 points)

$y=2,\ y=-1$



$$\int_{-1}^{2} y+2-y^2\,dy = \frac{1}{2}y^2+2y-\frac{1}{3}y^3\Big|_{-1}^{2} = \frac{9}{2}$$

2. Use the shell or washer method (as appropriate) to find the volume of the solid bounded by the graphs $f(x) =$ [redacted], and $g(x) = -(x+4)^2$ [redacted] and revolved around the x-axis. (10 points)

X +6

$X = -(x-4)^2+6 \rightarrow x-6 = -(x-4)^2$

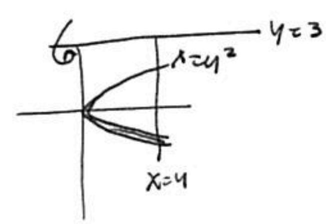
$x-6 = -x^2+8x-16$

$x^2-7x+10=0 \quad (x-2)(x-5)=0 \quad x=2, x=5$

$$\pi\int_{2}^{5} (x)^2 - \left(-(x-4)^2+6\right)^2 dx = \pi\int_{2}^{5} (x-4)^4 - 12(x-4)^2+36 - x^2\,dx$$

$$\pi\left[\frac{1}{4}(x-4)^5 - 4(x-4)^3 + 36x - \frac{1}{3}x^2\Big|_{2}^{5} = \pi\left[\frac{198}{5}\right]\right.$$

3. Use the shell method to find the volume of the solid bounded by the graphs $f(y) = y^2$, $x = 4$, around the line $y = 3$. (10 points)



$$2\pi\int_{-2}^{2}(4-y^2)(3-y)\,dy = 2\pi\int_{-2}^{2} 12-4y+3y^2+y^3\,dy =$$

$$2\pi\left[12y-2y^2-y^3+\frac{1}{4}y^4\Big|_{-2}^{2} = 2\pi[32] = 64\pi\right.$$

4. Find the length of arc, and the surface area when the function $y = x^2$ is revolved around the x-axis on the interval 1≤x≤2. (8 points)

$$S = \int_1^2 \sqrt{1+4x^2}\,dx \approx 3.1678409...$$

$$S = 2\pi\int_1^2 x^2\sqrt{1+4x^2}\,dx = 49.41623...$$

5. On the interval [4,9], find the average value of the function $f(x) = \sqrt{x}$. (8 points)

$$\bar{f} = \frac{1}{5}\int_4^9 \sqrt{x}\,dx = \frac{1}{5}\cdot\frac{2}{3}x^{3/2}\Big|_4^9 = \frac{2}{15}[27-8] = \frac{38}{15}$$

$$\approx 2.53\bar{3}$$

6. Find the area of the surface of revolution generated by $y = 9 - x^2$ on the interval [0, 3] revolved around the y-axis. (10 points)

$-2x$

$u = 1+4x^2$

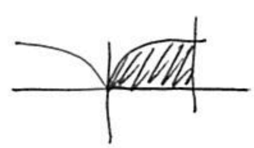
$du = 8x\,dx$

$$2\pi\int_0^3 x\sqrt{1+4x^2}\,dx$$

$$2\pi\cdot\frac{1}{8}\int u^{1/2}\,du = \frac{\pi}{4}\cdot\frac{2}{3}u^{3/2} \to \frac{\pi}{\cancel{4}_2}\cdot\frac{\cancel{2}}{3}\left(\sqrt{1+4x^2}\right)^3\Big|_0^3 =$$

$$\frac{\pi}{6}\left[(\sqrt{37})^3 - 1\right]$$

7. Find the centroid of a region of uniform density bounded by the graphs $y = x^{2/3}, y = 0, x = 8$. (16 points)



$$M = \int_0^8 x^{2/3}\,dx = \frac{3}{5}x^{5/3}\Big|_0^8 = \frac{3}{5}\cdot 32 = \frac{96}{5}$$

$$M_x = \frac{1}{2}\int_0^8 x^{4/3}\,dx = \frac{1}{2}\cdot\frac{3}{7}x^{7/3}\Big|_0^8 = \frac{3}{14}[128] = \frac{192}{7}$$

$$M_y = \int_0^8 x^{5/3}\,dx = \frac{3}{8}x^{8/3}\Big|_0^8 = \frac{3}{8}\cdot 256 = 96$$

$$\frac{M_y}{M} = \bar{x} = \frac{96}{96/5} = 5 \qquad \bar{y} = \frac{M_x}{M} = \frac{192/7}{96/5} = \frac{192}{7}\cdot\frac{5}{96} = \frac{10}{7}$$

$$\left(5, \tfrac{10}{7}\right)$$

8. A force of 20 lbs. stretches a spring 9 inches in an exercise machine. A) Find the work done in stretching the spring 12 inches. B) Find the work done stretching the spring an additional three inches (after completing part A). (10 points)

$F = kx \rightarrow 20 = k(3/4)$ in feet $\rightarrow k = \frac{80}{3}$

$W = \int_0^1 \frac{80}{3} x\, dx \quad \frac{40}{3}x^2 \Big|_0^1 = \frac{40}{3}$

$W = \int_1^{1.25} \frac{80}{3} x\, dx = \frac{40}{3} x^2 \Big|_1^{1.25} = \frac{40}{3}(1.25)^2 - \frac{40}{3} = \frac{15}{2}$

9. For the following integrals, state which method you would use, and which basic integration rule. Do not actually perform the integration. Methods may include: substitution, change of variables, complete the square, add/subtract, trig identities, long division, partial fractions, by parts, trig substitution, etc. Basic integration rules may include: power rule, log rule, exponential rule, trig functions, inverse trig functions, etc. Some problems may require more than more method or rule. (5 points each)

a. $\int \sin^4 x \cos^2 x\, dx$ trig identities, u-sub, trig functions, power rule

b. $\int x^2 \sin x^3\, dx$ u-sub, trig functions

c. $\int \arcsin x\, dx$ by parts, inverse trig, u-sub

d. $\int \frac{x^3}{\sqrt{16-x^2}} dx$ trig substitution, trig identities

e. $\int \frac{1}{(x^2+2x+11)^{\frac{3}{2}}} dx$ completing square, trig sub, trig functions

f. $\int \frac{x+4}{8x^3-1} dx$ partial fractions, log rules

10. Use Trapezoidal Rule to approximate the area under the curve of $\int_1^2 \frac{\ln(x)}{x+1} dx$ for n=6. (10 points)

$\Delta x = \frac{2-1}{6} = \frac{1}{6}$

$\int_1^2 f(x)dx \approx \frac{1}{12}\left[f(1)^{0} + 2f(1\tfrac{1}{6}) + 2f(\tfrac{8}{6}) + 2f(\tfrac{9}{6}) + 2f(\tfrac{10}{6}) + 2f(\tfrac{11}{6}) + f(2) \right] \approx$

$0.1462732084\ldots$

Part 2: In this section you will record your answers on paper along with your work. After scanning, submit them to a Canvas dropbox as directed. These questions will be graded by hand.

11. Use the definition of the hyperbolic secant function to prove that its derivative is the hyperbolic secant function times the hyperbolic tangent function, i.e. that $\frac{d}{dx}[\operatorname{sech} x] = -\operatorname{sech} x \tanh x$. (15 points)

$$\operatorname{sech} x = \frac{1}{\cosh x} = (\cosh x)^{-1} \qquad \frac{d}{dx}[\operatorname{sech} x] = -1(\cosh x)^{-2} \cdot \sinh x =$$

$$-1\left(\frac{1}{\cosh^2 x}\right) \cdot \sinh x = \frac{-1}{\cosh x} \cdot \frac{\sinh x}{\cosh x} = -\operatorname{sech} x \tanh x$$

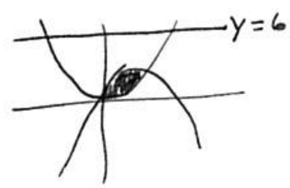
12. Use the disk or washer method to find the volume of revolution of the region bounded by the graphs $y = x^2, y = 4x - x^2$ around the line $y = 6$. (15 points)

$$x^2 = 4x - x^2 \qquad \pi \int_0^2 (4x - x^2 - 6)^2 - (x^2 - 6)^2 \, dx$$

$$0 = 4x - 2x^2$$

$$0 = 2x(2 - x)$$

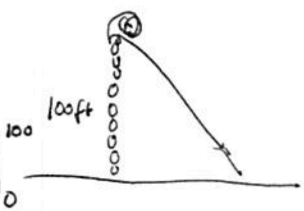
$$x = 0, \; x = 2 \qquad = \pi \cdot \frac{64}{3}$$



13. Find the work done by winding up a hanging cable of length 100 feet, and weight density 5 lbs/ft. (20 points)

$$\int_0^{100} 5(100 - x) \, dx = 5\left[100x - \frac{1}{2}x^2\right]_0^{100}$$

$$= 25{,}000$$



14. Set up (but do not solve) this rational expression $\frac{2x^4-5x^2+11x-}{(x^2+2)(x-3)^3(x^2+4)^2(x+1)}$ for decomposition by partial fractions. (16 points)

$$\frac{Ax+B}{x^2+2} + \frac{C}{x-3} + \frac{D}{(x-3)^2} + \frac{E}{(x-3)^3} + \frac{Fx+G}{x^2+4} + \frac{Hx+I}{(x^2+4)^2} + \frac{J}{x+1}$$

15. Integrate by an appropriate method. (10 points each)

a. $\int e^{-2x} \sin 9x\, dx$

$u = \sin 9x \qquad dv = e^{-2x}dx$

$du = 9\cos 9x \qquad v = -\frac{1}{2}e^{-2x}$

$-\frac{1}{2}e^{-2x}\sin 9x + \frac{9}{2}\int e^{-2x}\cos 9x\, dx$

$u = \cos 9x \qquad dv = e^{-2x}dx$

$du = -9\sin 9x\, dx \qquad v = -\frac{1}{2}e^{-2x}$

$-\frac{1}{2}e^{-2x}\sin 9x + \frac{9}{2}\left[-\frac{1}{2}\cos 9x\, e^{-2x} - \int \frac{9}{2}e^{-2x}\sin 9x\, dx\right]$

$\int e^{-2x}\sin 9x\, dx = -\frac{1}{2}e^{-2x}\sin 9x - \frac{9}{4}\cos 9x\, e^{-2x} - \int \frac{81}{4}e^{-2x}\sin 9x\, dx$

$\frac{85}{4}\int e^{-2x}\sin 9x\, dx = -\frac{1}{2}e^{-2x}\sin 9x - \frac{9}{4}e^{-2x}\cos 9x \rightarrow \boxed{-\frac{2}{85}e^{-2x}\sin 9x - \frac{9}{85}e^{-2x}\cos 9x + C}$

b. $\int \ln x\, dx$

$u = \ln x \qquad dv = dx$

$\frac{1}{x}dx = du \qquad v = x$

$x\ln x - \int 1\, dx = x\ln x - x + C$

c. $\int \sec^4 3x \tan 3x\, d\theta$

$\sec^3 3x\,(\sec 3x \tan 3x)$

$u = \sec 3x$

$du = 3\sec 3x \tan 3x\, dx$

$\frac{1}{3}\int u^3\, du = \frac{1}{12}u^4 + C$

$\frac{1}{12}\sec^4 3x + C$

d. $\int \operatorname{sech}^4 2x\,dx$ $\int \operatorname{sech}^2 2x\,(1+\tanh^2 2x)\,dx$ $u = \tanh 2x$

$du = 2\operatorname{sech}^2 2x\,dx$

$\frac{1}{2}\int 1+u^2\,du = \frac{1}{2}u + \frac{1}{6}u^3 + C$

$\frac{1}{2}\tanh 2x + \frac{1}{6}\tanh^3 2x + C$

e. $\int \frac{2x^2}{\sqrt{1+x^2}}dx$ $\tan\theta = x$ $\sqrt{1+x^2} = \sec\theta$ $dx = \sec^2\theta\,d\theta$

$\int \frac{2\tan^2\theta \cdot \sec^{\cancel{2}}\theta\,d\theta}{\cancel{\sec\theta}} = \int 2\tan^2\theta\sec\theta\,d\theta = 2\int(\sec^2\theta-1)(\sec\theta)\,d\theta$

$2\int \sec^3\theta - \sec\theta\,d\theta = 2\left[\frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta+\tan\theta| - \ln|\sec\theta+\tan\theta|\right] + C$

$= \sec\theta\tan\theta - \ln|\sec\theta+\tan\theta| + C$

$= x\sqrt{1+x^2} - \ln|\sqrt{1+x^2}+x| + C$

f. $\int \frac{4x^2}{x^3+x^2-x-1}dx = \int \frac{4x^2}{(x^2-1)(x+1)}dx = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x-1} \rightarrow A(x^2-1) + B(x-1) + C(x^2+2x+1)$

$x^2(x+1)-(x+1)$ $(x+1)^2(x-1)$

$Ax^2 - A + Bx - B + Cx^2 + 2Cx + C = 4x^2$

$A + C = 4$ $A = 3$

$B + 2C = 0$ $B = -2$

$-A - B + C = 0$ $C = 1$

$\int \frac{3}{x+1} + \frac{-2}{(x+1)^2} + \frac{1}{x-1}\,dx = 3\ln|x+1| + \frac{1}{x+1} + \ln|x-1| + C$

16. Determine whether or not the integral converges or diverges. If the integral converges, state its value. (16 points)

$$\int_0^2 \frac{1}{\sqrt{4-x^2}}dx$$

$\lim_{b\to 2}\int_0^b \frac{1}{\sqrt{4-x^2}}dx = \lim_{b\to 2} \arcsin\frac{x}{2}\Big|_0^b = \arcsin 1 - \cancel{\arcsin 0}^{\,0} = \frac{\pi}{2}$