4/3/2024

Integration Techniques:
Substitution (undoing the chain rule – 3.4)
Integration by parts (undoing the product rule – 3.5)
Applications?

Substitution is a technique for undoing a derivative that resulted from a chain rule.

Recall the chain rule: $F(x) = f\big(g(x)\big)$

$$F'(x) = f'\big(g(x)\big)g'(x)$$

Example:

$$h(x) = (x^3 + 2x)^5$$
$$h'(x) = 5(x^3 + 2x)^4(3x^2 + 2)$$

Now, we want to do the reverse…

$$\int 5(x^3 + 2x)^4(3x^2 + 2)dx$$

How do we recognize the pieces that are the derivative of the inside and the outside of the composite function so that we can get back to the original function?

This process involves finding a substitution for the "inside" (we call it u) and then seeing if some other piece of the integral corresponds to the derivative of that inside piece.

Let's let $u = x^3 + 2x$
Check then, what is du?, and is it (or a constant multiple of it) in the expression?

$$du = (3x^2 + 2)dx$$

Make the substitution if it matches

$$\int 5u^4 du$$

This is now just a power rule, one of our basic formulas:

$$\int 5u^4 du = 5\left(\frac{u^5}{5}\right) + C = u^5 + C$$

Substitute back in the original x-variable expression

$$(x^3 + 2x)^5 + C$$

Example.

$$\int \frac{x}{\sqrt{4 - x^2}} dx = \int (4 - x^2)^{-\frac{1}{2}} x dx$$

Is there any piece of this function that is a composition of functions? A function inside a power? A function in the exponent of an exponential? A function inside a log expression?

We have a function inside a power… the denominator is $(stuff)^{1/2}$, as a product $(stuff)^{-\frac{1}{2}}$.

The stuff on the inside is $4 - x^2 = u$
What is the derivative? $du = -2xdx$ … move the -2 to the other side of the equation by dividing

$$-\frac{1}{2}du = xdx$$

$$\int -\frac{1}{2}(u)^{-\frac{1}{2}}du = -\frac{1}{2}\left(u^{\frac{1}{2}}\right)\left(\frac{2}{1}\right) + C = -\sqrt{u} + C = -\sqrt{4 - x^2} + C$$

Always put the original variable back!!

Example.

$$\int \frac{x^2}{x^3 + 5}dx \quad vs. \int \frac{x^3 + 5}{x^2}dx$$

$$\int \frac{x^2}{x^3 + 5}dx = \int (x^3 + 5)^{-1}x^2dx$$

This will be (probably) a log function problem when the denominator is a higher degree than the denominator.

Compare to $\int \frac{x^2}{(x^3+5)^2}dx$ it would be more clear that this is a substitution, and for a power rule.

Try to let $u =$ denominator, then see if du is in the expression

$$u = x^3 + 5, du = 3x^2dx \to \frac{1}{3}du = x^2dx$$

$$\int \frac{1}{3}u^{-1}du = \frac{1}{3}\int \frac{1}{u}du = \frac{1}{3}\ln|u| + C = \frac{1}{3}\ln|x^3 + 5| + C$$

Other one is not substitution at all: du or the derivative CANNOT be in the denominator.

$$\int \frac{x^3 + 5}{x^2}dx = \int \frac{x^3}{x^2} + \frac{5}{x^2}dx = \int x + 5x^{-2}dx = \frac{1}{2}x^2 - \frac{5}{x} + C$$

Example.

$$\int e^{10x}dx$$

Let the exponent be the u

$$u = 10x, du = 10dx \rightarrow \frac{1}{10}du = dx$$

$$\int e^u \left(\frac{1}{10}\right) du = \frac{1}{10}\int e^u du = \frac{1}{10}e^u + C = \frac{1}{10}e^{10x} + C$$

Any time you see a natural log in an integral it has to be replaced with u.

Example.

$$\int \frac{1}{x\ln x}dx = \int (\ln x)^{-1}\left(\frac{1}{x}\right)dx \;\; or \;\; \int \frac{(\ln x)^2}{x}dx = \int \frac{\ln^2 x}{x}dx = \int (\ln x)^2\left(\frac{1}{x}\right)dx$$

$$u = \ln x$$
$$du = \frac{1}{x}dx$$

$$\int u^{-1}du = \int \frac{1}{u}du = \ln|u| + C = \ln(\ln x) + C$$

$$\int u^2 du = \frac{1}{3}u^3 + C = \frac{1}{3}(\ln x)^3 + C$$

Do you have "stuff" raised to a power? (under a root?) the inside is u
Do you have "stuff" in the exponent of an e? the exponent is u
Do you have a natural log in the problem? The log is u
Do you have a denominator (higher degree than the numerator), then the denominator is u.

Example.

$$\int_0^1 (3x-1)^4 dx$$

$$u = 3x - 1, du = 3dx \rightarrow \frac{1}{3}du = dx$$

Two strategies from this point:
You can continue as we did before, find the antiderivative in terms of u, convert back to x, and then plug in the original limits.
OR you can change the limits so that they match the values of u, and then not need to go back to x.

$$\int u^4\left(\frac{1}{3}\right)du = \frac{1}{3}\int u^4 du = \frac{1}{3}\left(\frac{u^5}{5}\right) = \frac{1}{15}(3x-1)^5$$

$$\int_0^1 (3x-1)^4 dx = \frac{1}{15}(3x-1)^5\Big|_0^1 = \left(\frac{1}{15}\right)[(3-1)^5 - (0-1)^5] = \frac{1}{15}[32+1] = \frac{33}{15}$$

Or, change the values of the limits to reflect the replacement variable:

$$u = 3x - 1$$

When x=0, u=3(0)-1= -1
When x=1, u=3(1)-1= 2

$$\int_{-1}^{2} \frac{1}{3} u^4 du = \frac{1}{3}\left(\frac{u^5}{5}\right)\Big|_{-1}^{2} = \frac{1}{15}(2^5 - (-1)^5) = \frac{1}{15}(32 + 1) = \frac{33}{15}$$

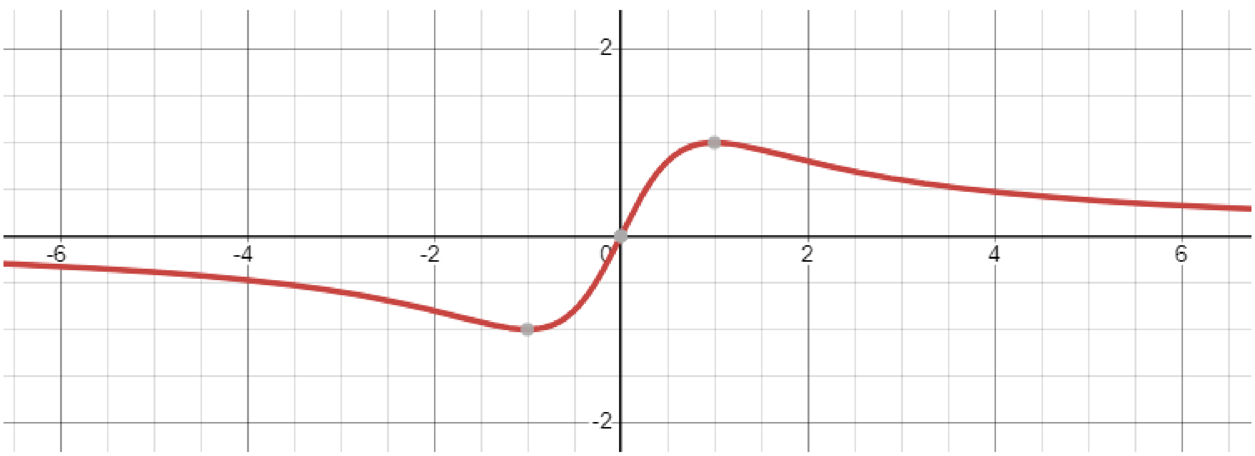
Example.

$$\int_{-2}^{2} \frac{2x}{1 + x^2} dx$$
$$u = 1 + x^2, du = 2xdx$$

$$\int_{5}^{5} \frac{1}{u} du = \ln|u||_5^5 = \ln(5) - \ln(5) = 0$$

When x= -2, $u = 1 + (-2)^2 = 5$
When x=2, u= $1 + 2^2 = 5$

What does the function $\frac{2x}{1+x^2}$ look like?



This is an odd function, and our properties of definite integrals are that $\int_{-a}^{a} f_{odd}(x)dx = 0$

$$\int \frac{1}{u} du = \ln|u| = \ln|1 + x^2|_{-2}^{2} = \ln|1 + (2)^2| - \ln|1 + (-2)^2| = \ln 5 - \ln 5 = 0$$

Integration by Parts
Used to undo product rule (part of the product rule)
The two functions being multiplied together are NOT the result of a chain rule.

$$\int udv = uv - \int vdu$$

The hope here is that the new integral is easier to integrate than the original.

We have to choose u and v for the problem, or u and dv

The general idea is that u should get easier when you take the derivative (alternatively, you may have to pick dv so that it gets easier when you integrate).

Consider the product rule:

$$(uv)' = u'v + v'u$$

$$\int (uv)' = uv = \int u'v + v'u = \int u'v + \int v'u$$

$$uv = \int u'v + \int v'u$$

$$\int v'u = \int udv = uv - \int u'v = uv - \int vdu$$

$$\int udv = uv - \int vdu$$

Example.

$$\int xe^x dx$$

$$u = x, dv = e^x dx$$
$$du = 1dx, v = e^x$$

$$xe^x - \int e^x dx = xe^x - e^x + C$$

Check:

$$(xe^x - e^x)' = e^x + xe^x - e^x = xe^x$$

Example.

$$\int 6x^2 \ln x \, dx$$

$$u = \ln x\,, dv = 6x^2 dx$$
$$du = \frac{1}{x} dx, v = 2x^3$$

$$2x^3 \ln x - \int \frac{2x^3}{x} dx = 2x^3 \ln x - \int 2x^2 dx = 2x^3 \ln x - \frac{2}{3}x^3 + C$$

Example.

$$\int \ln x \, dx$$

$$u = \ln x\,, dv = dx$$
$$du = \frac{1}{x} dx, v = x$$

$$x \ln x - \int \left(\frac{1}{x}\right) x\, dx = x \ln x - \int dx = x \ln x - x + C$$

The rule of thumb for picking u is LIATE: log, inverse trig, algebraic, trig, exponential

For us, the rule of thumb is just LAE: logs, algebraic, exponential

Algebraic: powers and roots and rational expressions…
Choose powers (positive integer powers, i.e. polynomials) to be u before anything else.

Example.

$$\int x\sqrt{3+x}dx$$

$$u = x, dv = (3+x)^{\frac{1}{2}}dx$$
$$du = dx, v = \int (3+x)^{\frac{1}{2}}dx = \frac{2}{3}(3+x)^{\frac{3}{2}}$$

$$v = \int (3+x)^{\frac{1}{2}}dx$$
$$w = 3 + x, dw = 1dx$$
$$= \int w^{\frac{1}{2}}dw = \frac{2}{3}w^{\frac{3}{2}} = \frac{2}{3}(3+x)^{\frac{3}{2}}$$

$$\frac{2}{3}x(3+x)^{\frac{3}{2}} - \int \frac{2}{3}(3+x)^{\frac{3}{2}}dx = \frac{2}{3}x(3+x)^{\frac{3}{2}} - \frac{2}{3}\left(\frac{2}{5}\right)(3+x)^{\frac{5}{2}} + C$$

Example.

$$\int x^3 e^{x^2} dx$$

Usual rule of thumb:

$$u = x^3, dv = e^{x^2}dx$$

The problem is I can't integrate e^{x^2} by itself.

I can integrate xe^{x^2} with substitution, so I'm going to borrow an x for dv

$$u = x^2, dv = xe^{x^2}$$
$$du = 2xdx, v = \int xe^{x^2}dx = \frac{1}{2}e^{x^2}$$

$$w = x^2, dw = 2xdx \rightarrow \frac{1}{2}dw = xdx$$

$$\int \frac{1}{2}e^w dw = \frac{1}{2}e^w$$

$$\frac{1}{2}x^2e^{x^2} - \int 2x\left(\frac{1}{2}\right)e^{x^2}dx = \frac{1}{2}x^2e^{x^2} - \int xe^{x^2}dx = \frac{1}{2}x^2e^{x^2} - \frac{1}{2}e^{x^2} + C$$

Sometimes we have to do this process more than once.

$$\int x^3e^xdx$$

$$u = x^3, dv = e^xdx$$
$$du = 3x^2dx, v = e^x$$

$$x^3e^x - \int 3x^2e^xdx =$$
$$u = 3x^2, dv = e^xdx$$
$$du = 6xdx, v = e^x$$

$$x^3e^x - \left[3x^2e^x - \int 6xe^xdx\right] = x^3e^x - 3x^2e^x + \int 6xe^xdx$$
$$u = 6x, dv = e^xdx$$
$$du = 6dx, v = e^x$$

$$x^3e^x - 3x^2e^x + 6xe^x - \int 6e^xdx = x^3e^x - 3x^2e^x + 6xe^x - 6e^x + C$$

Since I was multiplying by x^3, it took three applications of the integration by parts to get to the end.

Tabular method

$\pm$	u	dv
$+$	x^3	e^x
$-$	$3x^2$	e^x
$+$	$6x$	e^x
$-$	6	e^x
	0	e^x

$$= x^3e^x - 3x^2e^x + 6xe^x - 6e^x + C$$

Generally: let u be the polynomial, and dv has to integrate by itself, so eg. $e^x, \sqrt{3+x}, e^{4x}, \ldots$

Applications
We've discussed area under a curve (between the curve and the x-axis).
But we can also do the area between two curves.

$$\int_a^b f(x)_{top} - g(x)_{bottom}dx$$

Example.
Find the area between the curves $f(x) = x + 3, g(x) = x^2 - 4x + 3$ on the interval [0,3].

If you are given an interval, use that as your limits of integration. If not, then the intention is to find out where the graphs intersect with each other.

$$\int_0^3 (x+3)-(x^2-4x+3)dx = \int_0^3 -x^2+5xdx = -\frac{1}{3}x^3+\frac{5}{2}x^2\Big|_0^3 = -9+\frac{45}{2} = \frac{27}{2}$$

If no interval was given, then set the two equations equal to each other

$$x^2-4x+3 = x+3$$
$$x^2-5x = 0$$
$$x(x-5) = 0$$

$$x = 0, x = 5$$

These would be my limits of integration.

And we still have to think about the possibility that the functions change sequencing. Where for part of the interval f(x) is on top and g(x) on the bottom, and then switches to g(x) on top, and f(x) on the bottom.

May need to split the integral at the intersection and find the absolute value of the area before adding the pieces (for geometric area).

It's always a good idea to graph these problems before just mechanically integrating.

Average value of a function.

$$\bar{f}(x) = \frac{1}{b-a}\int_a^b f(x)dx$$

Integral is the total area, the (b-a) is the length of the region… area/length = average height. If I converted my area to a rectangle, how high would the rectangle have to be? = average values of the function on that interval.

$$\frac{1}{9-0}\int_0^9 5+\sqrt{t}dt = \frac{1}{9}\left[5t+\frac{2}{3}t^{\frac{3}{2}}\right]_0^9 = \frac{1}{9}\left[45+\frac{2}{3}(27)\right] = \frac{1}{9}[45+18] = 7$$

Partial Fractions: is a technique for dealing with rational functions whose denominators factor.