

4/24/2024

Double Integrals

When we integrate a function with two (or more) variables, we treat it similarly to the way we treat partial derivatives: we work with one variable at a time, and then treat any other variables as constant.

$$\int 3xy - e^x + \frac{1}{y} dx = (3y) \left(\frac{x^2}{2} \right) - e^x + \left(\frac{1}{y} \right) x + C(y) = \frac{3}{2} x^2 y - e^x + \frac{x}{y} + C(y)$$

$$\int 3xy - e^x + \frac{1}{y} dy = \frac{(3x)y^2}{2} - ye^x + \ln(y) + K(x)$$

$$\begin{aligned} \int_1^{x^3} 3xy - e^x + \frac{1}{y} dy &= \frac{(3x)y^2}{2} - ye^x + \ln(y) \Big|_{y=1}^{y=x^3} = \\ &= \frac{3x}{2} (x^3)^2 - x^3 e^x + \ln(x^3) - \left(\frac{3x}{2} - e^x + \ln(1) \right) = \\ &= \frac{3x^7}{2} - x^3 e^x + \ln(x^3) - \frac{3x}{2} + e^x \end{aligned}$$

If I have variables or constants in my limits, then I am reduced to a function of just one variable. The variable I integrated with disappears (since I can't have y in the limit when I integrate with y).

Double integrals

$$\int_a^b \int_{g(x)}^{h(x)} f(x, y) dy dx = \int_a^b \left[\int_{g(x)}^{h(x)} f(x, y) dy \right] dx$$

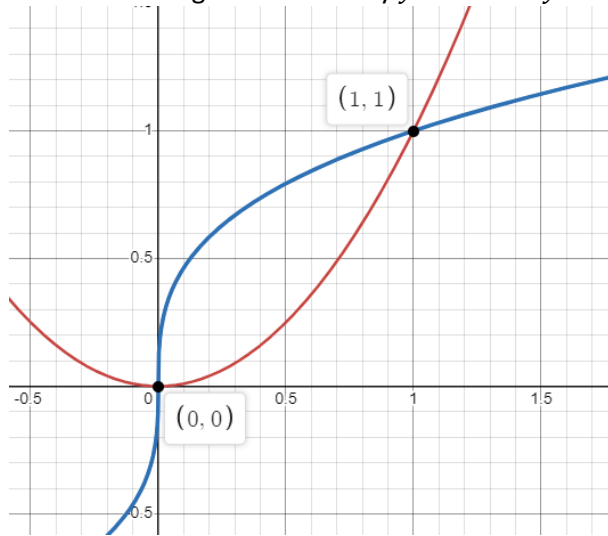
$$\int_c^d \int_{w(y)}^{v(y)} k(x, y) dx dy = \int_c^d \left[\int_{w(y)}^{v(y)} k(x, y) dx \right] dy$$

Example.

$$\begin{aligned} \int_0^4 \int_0^{2x} (x^2 y - y^2 + 3) dy dx &= \int_0^4 \left[\int_0^{2x} (x^2 y - y^2 + 3) dy \right] dx = \\ \int_0^4 \left[\frac{x^2 y^2}{2} - \frac{y^3}{3} + 3y \right]_0^{2x} dx &= \int_0^4 \frac{x^2 (2x)^2}{2} - \frac{(2x)^3}{3} + 3(2x) dx = \int_0^4 2x^4 - \frac{8x^3}{3} + 6x dx = \\ \frac{2}{5} x^5 - \frac{2}{3} x^4 + 3x^2 \Big|_0^4 &= \frac{2048}{5} - \frac{512}{3} + 48 = \frac{4304}{15} \end{aligned}$$

When we integrate in one variable, we can express that one variable integral for the area in terms of a double integral.

Consider the region bounded by $y = x^2$ and $y = \sqrt[3]{x}$.



$$\int_0^1 \sqrt[3]{x} - x^2 dx$$

$$\int_0^1 \int_{\text{bottom_function}}^{\text{top_function}} dy dx = \int_0^1 \int_{x^2}^{\sqrt[3]{x}} 1 dy dx = \int_0^1 y \Big|_{x^2}^{\sqrt[3]{x}} dx = \int_0^1 \sqrt[3]{x} - x^2 dx$$

The double integral, when I integrate just 1, then I get the area of the region in the plane (just the difference between two functions with a single integral). But if I integrate a function (other than 1), I get the volume, where the function is specifying the height of the region.

Double integral is finding the volume above a region in the plane.

Recall $z = f(x, y)$

When we switch the order of integration we have $\int_c^d \int_{\text{left_function}}^{\text{right_function}} f(x, y) dx dy$, this is like setting up our rectangles horizontally rather than vertically.

Consider changing the limits of integration for:

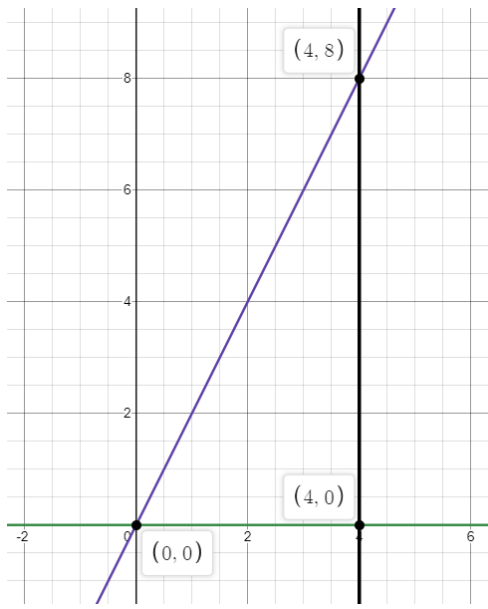
$$\int_0^4 \int_0^{2x} (x^2 y - y^2 + 3) dy dx$$

$$f(x, y) = x^2 y - y^2 + 3$$

The surface $z = f(x, y)$ above the region in the plane that defines the height of the region.

The limits of the double integral are the region in the x-y plane:

$$y = 0, y = 2x, x = 0, x = 4$$



$y=2x$ for $x \dots x=y/2$

$$\int_0^8 \int_{\frac{y}{2}}^4 x^2 y - y^2 + 3 \, dx dy = \int_0^8 \left[\frac{x^3 y}{3} - xy^2 + 3x \right]_{\frac{y}{2}}^4 dy =$$

$$\int_0^8 \frac{64y}{3} - 4y^2 + 12 - \left(\frac{\left(\frac{y}{2}\right)^3 y}{3} - \left(\frac{y}{2}\right) y^2 + 3\left(\frac{y}{2}\right) \right) dy =$$

$$\int_0^8 \frac{64}{3} y - 4y^2 + 12 - \frac{y^4}{24} + \frac{y^3}{2} - \frac{3y}{2} dy = \int_0^8 -\frac{y^4}{24} + \frac{y^3}{2} - 4y^2 + \frac{119}{6} y + 12 dy =$$

$$-\frac{y^5}{120} + \frac{y^4}{8} - \frac{4}{3} y^3 + \frac{119}{12} y^2 + 12y \Big|_0^8 = -\frac{4096}{15} + 512 - \frac{2048}{3} + \frac{1904}{3} + 96 = \frac{4304}{15}$$

In a double integral, you can switch the order of integration (sometimes you need to in order to do the integration), but you need to reset up the limits of integration from the graph of the region in the plain. You can only directly swap the order when both sets of limits are constant. The outer integral always has to have constants. In principle, it should not matter what order you set the integral up in, the value should be in both cases.

Probability in 2-variables

Consider the probability density function: $f(x, y) = kx^2 y$ on the region bounded by $0 \leq x \leq 2$, $0 \leq y \leq x$.

$$\int_0^2 \int_0^x kx^2 y dy dx = 1$$

$$\int_0^2 \int_0^x kx^2 y dy dx = \int_0^2 kx^2 \left(\frac{y^2}{2} \right) \Big|_0^x dx = \int_0^2 \frac{k}{2} x^4 dx = \frac{k}{2} \left(\frac{x^5}{5} \right) \Big|_0^2 = \frac{k}{10} (32) = \frac{16k}{5} = 1$$

$$k = \frac{5}{16}$$

$$f(x, y) = \frac{5}{16}x^2y$$

$$P(0 \leq x \leq 1, y \leq x) = \int_0^1 \int_0^x \frac{5}{16}x^2y dy dx$$

Mean: $\mu, E(X)$ (expected value)

$$\mu_x = \int_0^2 \int_0^x x \left(\frac{5}{16}x^2y \right) dy dx$$

$$\mu_y = \int_0^2 \int_0^x y \left(\frac{5}{16}x^2y \right) dy dx$$

Compare e^{x^2} to e^{-4x}

$$\frac{d}{dx}[e^{x^2}] = 2xe^{x^2}$$

$$\frac{d}{dx}[e^{-4x}] = -4e^{-4x}$$

$$\left(-\frac{1}{4}\right)[(-4)e^{-4x}]$$