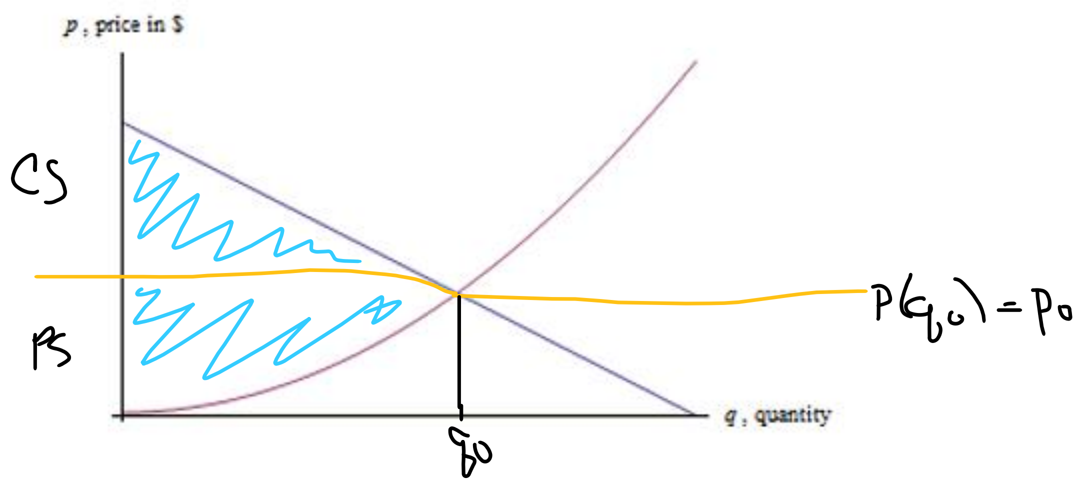
4/17/2024

Consumer and Producer's Surplus
Continuous Income Streams
Exponential Growth and Decay
Probability Distributions (one variable)

Consumer's and Producer's Surplus

Demand curve, and a supply curve.
Related to the law of supply and demand.
When supply and demand are balanced they are in equilibrium.
When they are not in equilibrium, there is a gap between what is available and what is in demand.



To calculate the Consumer's and Producer's surplus:
Given $p = d(q), p = s(q)$.

1) We need to find the equilibrium of p and q for the supply and demand functions (solve as a system of simultaneous equations)
   We can do this algebraically or graphically (sometimes these will be linear systems and sometimes they will be nonlinear systems).

Consumer's surplus

$$CS = \int_0^{q_0} [d(q) - p(q_0)]dq$$

Producer's surplus

$$PS = \int_0^{q_0} [p(q_0) - s(q)]dq$$

Example.

Suppose for the demand equation we have $p = d(q) = 150 - 0.8q$, and the supply equation for the same product is $p = s(q) = 5.2q$

Find the producer's and consumer's surplus

Step 1. Find the equilibrium. Find the values of p and q that balance the two equations. (intersection point)

$$p = 5.2q$$
$$p = 150 - 0.8q$$

$$5.2q = 150 - 0.8q$$
$$6q = 150$$
$$q_0 = 25$$
$$p_0 = 130$$

$$CS = \int_0^{q_0} [d(q) - p(q_0)]dq = \int_0^{25} (150 - 0.8q - 130)dq = \int_0^{25} 20 - 0.8q\, dq = 20q - 0.4q^2|_0^{25} =$$

$$20(25) - 0.4(25)^2 = 250$$

$$PS = \int_0^{q_0} [p(q_0) - s(q)]dq = \int_0^{25} 130 - (5.2q)dq = 130q - 2.6q^2|_0^{25} =$$

$$130(25) - 2.6(25^2) = 1625$$

Example.

The tables below show information about the demand and supply functions for a product. For both functions, q is the quantity and p is the price, in dollars.

q	0	100	200	300	400	500	600	700
p	70	61	53	46	40	35	31	28

← demand

q	0	100	200	300	400	500	600	700
p	14	21	28	33	40	47	54	61

← supply

Estimate the demand and supply equations with a linear function.

$$m = \frac{28 - 70}{700 - 0} = -\frac{42}{700} = -0.06$$

$$p - 70 = -0.06(q - 0)$$
$$p = 70 - 0.06q$$

$$m = \frac{61 - 14}{700 - 0} = \frac{47}{700}$$

$$p = 14 + \frac{47}{700}q$$

$$CS = \int_0^{400} [70 - 0.06q - 40]dq = \int_0^{400} 30 - 0.06q\, dq = 30q - 0.03q^2|_0^{400} =$$
$$30(400) - (0.03)(400^2) = 7200$$

$$PS = \int_0^{400} \left[40 - \left(14 + \frac{47}{700}q\right)\right] dq = \int_0^{400} 26 - \frac{47}{700}q\, dq = 26q - \frac{47}{1400}q^2\Big|_0^{400} =$$

$$26(400) - \frac{47(400^2)}{1400} \approx 5028.57$$

Example.
Suppose the demand equation for a product is given by $p = d(q) = 150 - 0.5q$, and the supply equation is given by $p = s(q) = 0.002q^2 + 1.5$.

Find the intersection of the two curves.

$$150 - 0.5q = 0.002q^2 + 1.5$$
$$0.002q^2 + 0.5q - 148.5 = 0$$

X500 to get rid of decimals

$$q^2 + 250q - 74{,}250 = 0$$

$$q = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-250 \pm \sqrt{250^2 - 4(-74250)}}{2} = \frac{-250 \pm \sqrt{359{,}500}}{2} \to \frac{-250 + \sqrt{359{,}500}}{2}$$

$$\approx 174.79$$

$$\to q_0 = 175\ (integer)$$

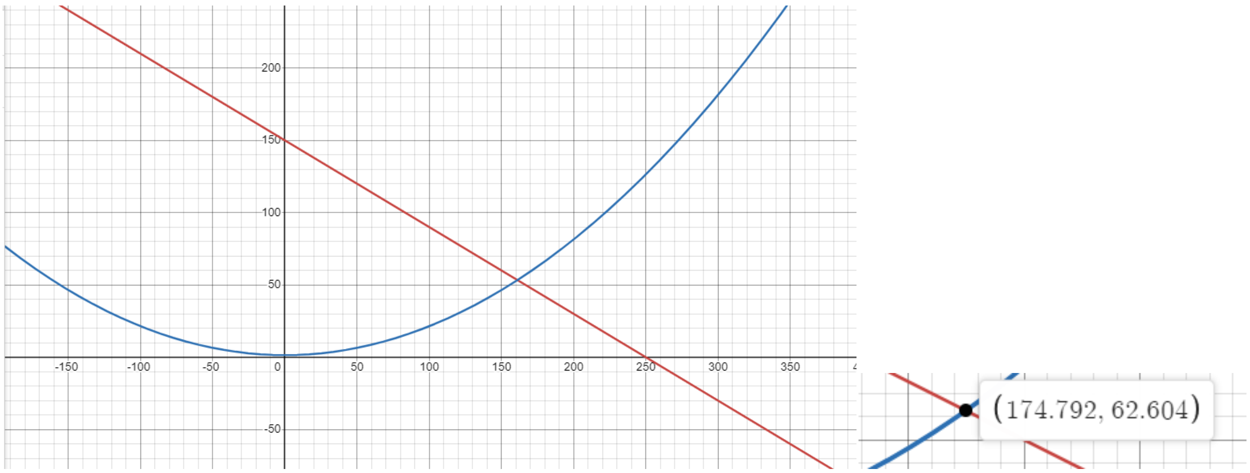
$$p_0 = 150 - 0.5(175) = 62.5$$

$$CS = \int_0^{175} [150 - 0.5q - 62.5]dq = \int_0^{175} 87.5 - 0.5q\, dq = 87.5q - 0.25q^2|_0^{175} =$$

$$87.5(175) - 0.25(175)^2 = 7656.25$$

$$PS = \int_0^{175} [62.5 - (0.002q^2 + 1.5)]dq = \int_0^{175} 61 - 0.002q^2\, dq = 61q - \frac{0.002}{3}q^3\Big|_0^{175} =$$

$$61(175) - \frac{0.002(175)^3}{3} = 7102.08$$



Continuous Income Stream
$F(t)$ is a stream of continuous income being deposited into an account that earn interest compounded continuously.

The present value of the account after T years:

$$PV = \int_0^T F(t)e^{-rt}dt$$

You have an opportunity to buy a business that will earn $75,000 per year continuously over the next eight years. Money can earn 2.8% per year, compounded continuously. Is this business worth its purchase price of $630,000?

$$F(t) = \$75{,}000$$

$$\int_0^8 75000e^{-0.028t}dt = \frac{75000}{-0.028}e^{-0.028t}\Big|_0^8 = \frac{75000}{-0.028}e^{-0.028(8)} + \frac{75000}{0.028}e^{-0.028(0)} =$$

$$2678571.429 - 2141022.681 = 537{,}548.747\ldots$$

This is worth less than the value of the business and so it is not a good investment.

In the homework section of the textbook, they do include example problems where F(t) is a function of time and not a constant. You will need integration techniques like integration by parts to solve the integral.

If you get an equation that cannot be solved by substitution or by integration by parts, you can use your calculator do the integration numerically.

MATH→#9 fnInt(
fnInt(function,variable, lower bound, upper bound)

Growth/Decay problems

A bank pays 2% interest on its certificate of deposit accounts, but charges a \$20 annual fee. If you initially invest \$3,000, how much will you have after 10 years?

You may recognize this as the example from the beginning of the section, for which we set up the equation $B'(t) = 0.02B(t) - 20$, or more simply, $\frac{dB}{dt} = 0.02B - 20$

Interest alone:

$$B'(t) = \frac{dB}{dt} = 0.02B$$
$$B = B_0 e^{0.02t}$$

$$\frac{dB}{dt} = 0.02B$$

$$\int \frac{dB}{B} = \int 0.02dt$$
$$\ln B = 0.02t + C$$

$$B(t) = B_0 e^{0.02t}$$

With the fee

$$\frac{dB}{dt} = 0.02B - 20$$

$$\frac{dB}{dt} = 0.02(B - 1000)$$

$$\int \frac{dB}{B - 1000} = \int 0.02dt$$

$$\ln|B - 1000| = 0.02t + C$$
$$B - 1000 = B_0 e^{0.02t}$$

$$B(t) = 1000 + B_0 e^{0.02t}$$

$$3000 = 1000 + B_0 e^0$$
$$B_0 = 2000$$

$$B(t) = 1000 + 2000e^{0.02t}$$

How much in the account after 10 years:

$$B(10) = 1000 + 2000e^{0.02(10)} = \$3442.81$$

Limited Growth problems have a maximum possible to be achieved… in population models this is referred to as a carrying capacity.

Model:

$$\frac{dy}{dt} = ky(M - y)$$

$$\frac{dy}{dt} = ky\left(1 - \frac{y}{M}\right)$$

Solution:

$$y(t) = \frac{M}{1 + Ae^{kt}}$$

A colony of 100 rabbits is introduced to a reclaimed forest. After 1 year, the population has grown to 300. It is estimated the forest can sustain 5000 rabbits. The forest service plans to reintroduce wolves to the forest when the rabbit population reaches 3000 rabbits. When will that occur?

$$y(t) = \frac{5000}{1 + Ae^{k(t)}}$$
$$y(0) = 100 = \frac{5000}{1 + Ae^{k(0)}}$$
$$100 = \frac{5000}{1 + A}$$
$$100(1 + A) = 5000$$
$$100 + 100A = 5000$$
$$100A = 4900$$
$$A = 49$$

$$y(t) = \frac{5000}{1 + 49e^{kt}}$$

$$y(1) = 300 = \frac{5000}{1 + 49e^{k}}$$

$$300(1 + 49e^{k}) = 5000$$
$$300 + 14700e^{k} = 5000$$
$$14700e^{k} = 4700$$
$$e^{k} = \frac{4700}{14700}$$

$$\ln e^{k} = k = \ln\left(\frac{47}{147}\right) \approx -1.14028 \ldots$$

$$y(t) = \frac{5000}{1 + 49e^{-1.14028t}}$$

$$\frac{5000}{1 + 49e^{-1.14028t}} = 3000$$

$$3000(1 + 49e^{-1.14028t}) = 5000$$
$$3000 + 147000e^{-1.14028t} = 5000$$

$$147000e^{-1.14028t} = 2000$$
$$e^{-1.14028t} = \frac{2}{147}$$

$$\ln(e^{-1.14028t}) = -1.14028t = \ln\left(\frac{2}{147}\right)$$

$$t = \frac{\ln\left(\frac{2}{147}\right)}{\ln\left(\frac{47}{147}\right)} \approx 3.7686 \ldots years$$

Probability Distributions

Continuous probability distribution, these are defined by functions (curves) the probability is calculated by finding the area under the curve.

For example. The standard normal distribution is given by the probability density function (pdf) $f(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$

A more general normal distribution function:

$$f(x) = \frac{1}{\sqrt{2\pi\sigma}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

If we want to calculate the probability under the normal curve

$$P(X \leq z) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z} e^{-\frac{x^2}{2}}\,dx$$

The scaling factor involving pi comes from the need to have the area under the curve be equal to 1.

Example.

We have a random variable that is distributed on the interval [0,2], according to the probability density function given by $f(x) = kx^3$.

1) Find the value of k that makes this a legitimate probability density function.

$$\int_0^2 kx^3 dx = \frac{k}{4}x^4\Big|_0^2 = \frac{k}{4}(16) = 4k = 1$$
$$k = \frac{1}{4}$$

$$f(x) = \frac{1}{4}x^3$$

2) Find the probability within a given range of outcomes:
   What is probability of obtaining a value between 1 and 1.5? $P(1 \leq x \leq 1.5)$

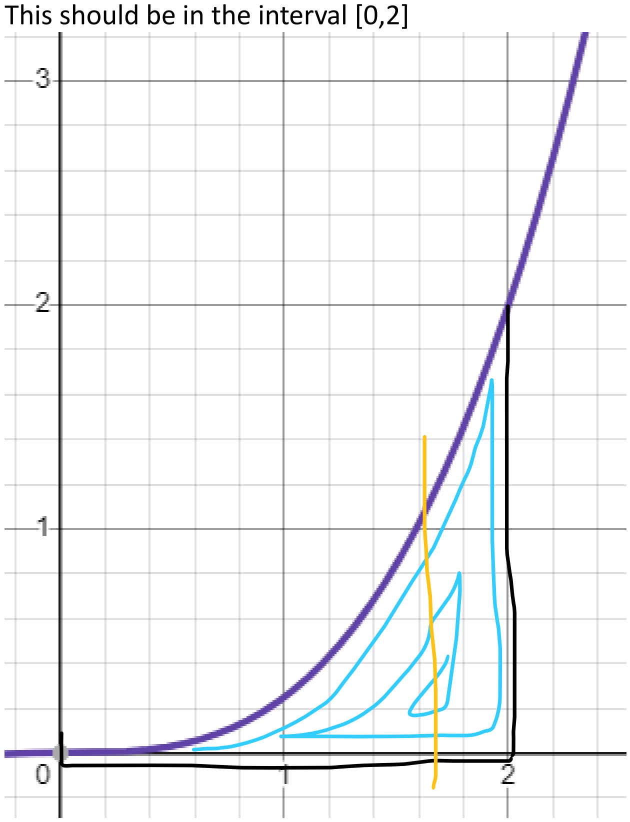
$$P(1 \le x \le 1.5) = \int_1^{1.5} \frac{1}{4}x^3 dx = \frac{1}{16}x^4\Big|_1^{1.5} = \frac{1}{16}[1.5^4 - 1^4] = \frac{1}{16}[5.0625 - 1] = 0.2539\ldots$$

3) Find the median of the distribution:

$$0.5 = \int_0^m \frac{1}{4}x^3 dx = \frac{1}{16}x^4\Big|_0^m = \frac{1}{16}[m^4]$$

$$\frac{1}{16}m^4 = 0.5$$
$$m^4 = 8$$
$$m = \sqrt[4]{8} \approx 1.68179\ldots$$

This should be in the interval [0,2]



4) The mean of a distribution:

$$\mu = \int_a^b xf(x)dx = \int_0^2 x\left(\frac{1}{4}x^3\right)dx = \int_0^2 \frac{1}{4}x^4 dx = \frac{1}{20}x^5\Big|_0^2 = \frac{1}{20}[32] = 1.6$$

5) The variance (and from that the standard deviation) of a distribution.

$$\sigma^2 = \int_a^b (x-\mu)^2 f(x)dx = \int_0^2 (x-1.6)^2\left(\frac{1}{4}x^3\right)dx = \frac{8}{75} \approx 0.1066666\ldots$$

$$\sigma = \sqrt{\frac{8}{75}} \approx 0.326598\ldots$$