3/6/2024

Absolute Extrema, Applications
Implicit Differentiation
Related Rates

Reminder about absolute extrema:
For absolute extrema: there is always an absolute max and absolute min on a function on any closed interval (an interval that includes the endpoints).

The procedure for finding them is to find any critical points on the function.
Check to see if the critical points are in the interval provided. Discard any outside the interval.
Check the values at the critical points (that remain) and the endpoints of the interval in the original function.
The largest result is the absolute maximum and the smallest result is the absolute minimum.

Extrema (max/min) they are y-values, and the x-values are where the extrema occur.
It is possible that the extreme value will occur at more than one place.

In this online textbook, they use the term "global" instead of "absolute".

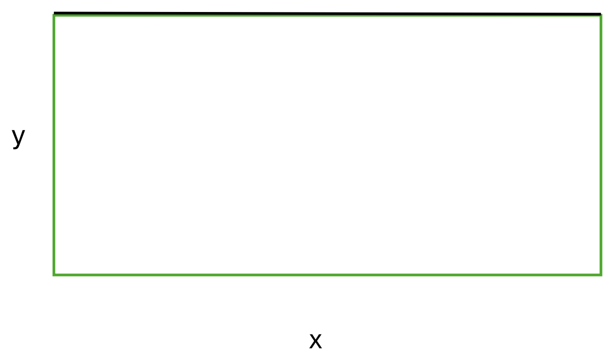
Example.
Find the absolute extrema for the function $f(x) = x^3 - 3x^2 - 9x + 5$ on the interval $-2 \leq x \leq 6$ ([-2,6])

$$f'(x) = 3x^2 - 6x - 9 = 0$$
$$3(x^2 - 2x - 3) = 0$$
$$(x - 3)(x + 1) = 0$$
$$x = 3, -1$$

Both are on the interval.

$$f(-2) = 3$$
$$f(-1) = 10$$
$$f(3) = -22 - absolute\ min$$
$$f(6) = 59 - absolute\ max$$

Optimization (applied)
Example.
The manager of a garden store wants to build a 600 square foot rectangular enclosure on the store's parking lot in order to display some equipment. Three sides of the enclosure will be built of redwood fencing, at a cost of $7 per running foot. The fourth side will be built of cement blocks, at a cost of $14 per running foot. Find the dimensions of the least costly such enclosure.



Fencing: $7y + 7x + 7y + 14x = 14y + 21x = F(x, y)$
Area: $600 = xy$
Solve the area equation for y and replace y in the fencing equation

$$y = \frac{600}{x}$$

$$F(x) = 14\left(\frac{600}{x}\right) + 21x = \frac{8400}{x} + 21x = 8400x^{-1} + 21x$$

$$F'(x) = -8400x^{-2} + 21 = 0$$

$$-\frac{8400}{x^2} + 21 = 0$$

$$\frac{8400}{x^2} = 21$$

$$8400 = 21x^2$$

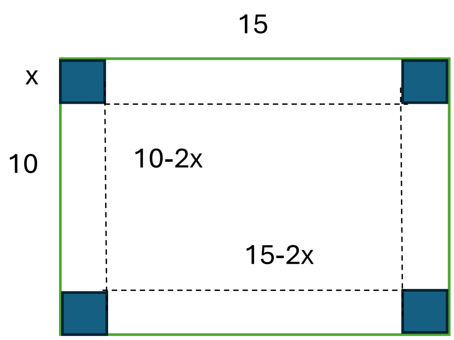
$$x^2 = 400$$
$$x = 20, -20$$

Since x is the length of a fence, it can't be negative.
The min cost will have to be at 20.

$$F''(x) = 16{,}800x^{-3}$$
$$F''(20) = \frac{16{,}800}{20^3} > 0$$

Confirms that this is a minimum.

Example.
You have a 10 inch by 15 inch piece of tin which you plan to form into a box (without a top) by cutting a square from each corner and folding up the sides. How much should you cut from each corner so the resulting box has the greatest volume?



Volume=$(10-2x)(15-2x)x$

$$V(x) = x(150 - 20x - 30x + 4x^2) = 4x^3 - 50x^2 + 150x$$

Take the derivative to find critical points for the optimum

$$V'(x) = 12x^2 - 100x + 150 = 2(6x^2 - 50x + 75)$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{50 \pm \sqrt{2500 - 4(6)(75)}}{2(6)} = \frac{50 \pm \sqrt{2500 - 1800}}{2(6)} = \frac{50 \pm \sqrt{700}}{12} =$$

$$x \approx 6.3714 \ldots, 1.96187 \ldots$$

$$15 - 2x = 15 - 2(6.37) \approx 2.26$$
$$10 - 2x = 10 - 2(6.37) \approx -2.74$$

I can't remove a square of size 6.37 from the side that is 10 inches long.

$$15 - 2x = 15 - 2(1.96) \approx 11.08$$
$$10 - 2x = 10 - 2(1.96) \approx 6.08$$

Maximum volume will be approximately (11.08)(6.08)(1.96) ≈ 132.04

Open-top box problems.
Applied optimization problems (in one variable)

Example.
A company sells q ribbon winders per year at $p per ribbon winder. The demand function for ribbon winders is given by: p=200-0.02q. The ribbon winders cost $30 apiece to manufacture, plus there are fixed costs of $9000 per year. Find the quantity where profit is maximized.

Cost = (cost per item)(number of items) + fixed cost
Revenue = (price per item)(number of items)
Profit = Revenue minus the cost

$$C(q) = 30q + 9000$$
$$R(q) = qp = q(200 - 0.02q) = 200q - 0.02q^2$$
$$P(q) = 200q - 0.02q^2 - 30q - 9000 = 170q - 0.02q^2 - 9000$$
$$P'(q) = 170 - 0.04q = 0$$

$$170 = 0.04q$$
$$q = 4250$$

The number of items sold generally needs to be an integer (whole number, and positive)

Marginal revenue and marginal cost… "marginal" is the derivative.
How much does the revenue or cost change from the present point to one unit more.

Average cost: the cost function divided by the number of items sold:

$$C_{avg}(x) = \bar{C}(x) = \frac{C(x)}{x}$$

$$C(q) = 30q + 9000$$

Average cost: $\frac{30q+9000}{q} = \bar{C}(x) = 30 + \frac{9000}{q}$
The marginal average cost is the derivative of the average cost.

Implicit Differentiation and Related Rates
Implicit Differentiation is just the chain.

Implicit differentiation is used to find the derivative (slope of the tangent) for implicitly defined functions, we can't solve for y easily.

$$x^2 + y^2 = 4$$

$$x^2 - xy + y^2 = \ln(y)$$

The idea here is that we "assume" that y is a function of x, and so when we take the derivative of y, we apply the chain rule:

$$\frac{d}{dx}(y) = y' = \frac{dy}{dx}$$

$$\frac{d}{dx}[f(y)] = f'(y)y'$$

Example.

$$x^2 + y^2 = 4$$

$$\frac{d}{dx}[x^2] + \frac{d}{dx}[y^2] = \frac{d}{dx}[4]$$

$$2x + 2y(y') = 0$$

Solve for y'

$$2yy' = -2x$$

$$y' = -\frac{2x}{2y} = -\frac{x}{y}$$

Equation was $x^2 + y^2 = 4, x = 1$

$$1^2 + y^2 = 4$$
$$y^2 = 3$$
$$y = \pm\sqrt{3}$$

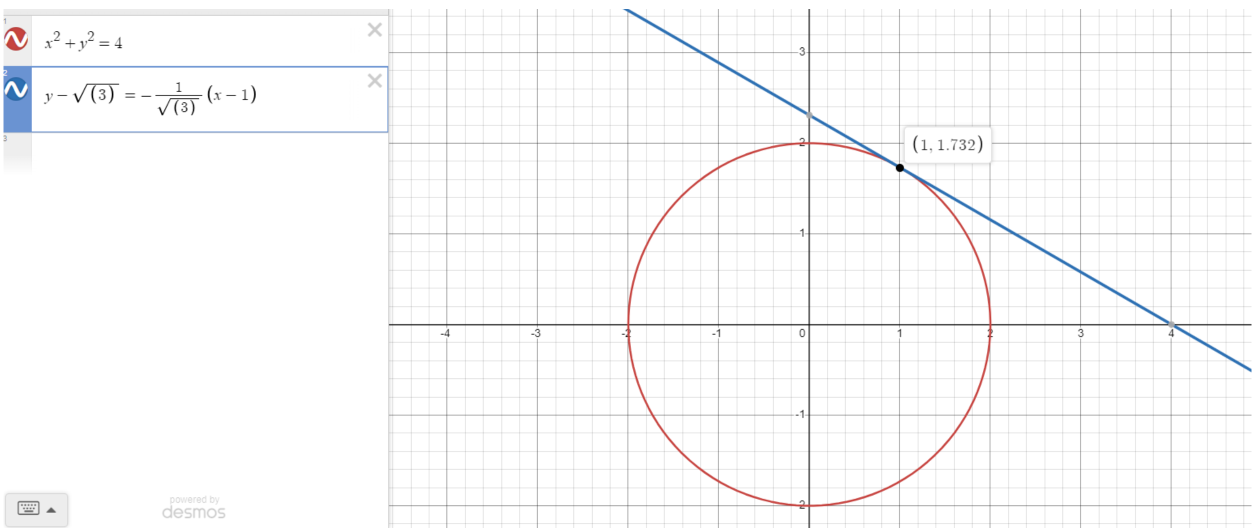
Consider the point $(1, \sqrt{3})$

What is the slope of the tangent at that point?

$$\frac{dy}{dx} = y' = -\frac{x}{y} = -\frac{1}{\sqrt{3}}$$

What is the equation of the tangent line at that same point?

$$y - \sqrt{3} = -\frac{1}{\sqrt{3}}(x - 1)$$



Example. Find the derivative of y with respect to x for the implicitly defined function

$$x^2 - xy + y^2 = \ln(y)$$

Since we are treating y as a function of x, when y is multiplied by x, we have to use a product rule.

$$\frac{d}{dx}[x^2] - \frac{d}{dx}[xy] + \frac{d}{dx}[y^2] = \frac{d}{dx}[\ln y]$$

$$2x - (1(y) + xy') + 2yy' = \frac{1}{y}y'$$

$$2x - y - xy' + 2yy' = \frac{y'}{y}$$

Collect the terms without y-prime on one side of the equation and terms with y-prime on the other side.

$$2x - y = \frac{y'}{y} + xy' - 2yy'$$

Factor out y-prime

$$2x - y = y'\left(\frac{1}{y} + x - 2y\right)$$

The divide by whatever is multiplying y-prime

$$\frac{2x - y}{\frac{1}{y} + x - 2y} = y'$$

Can simplify this a little bit by multiplying by the common denominator (y)

$$\frac{(2x - y)y}{\left(\frac{1}{y} + x - 2y\right)y} = \frac{2xy - y^2}{1 + xy - 2y^2} = y'$$

There is a faster way to obtain the implicit derivative using partial derivatives.

Create a function of two variables $F(x, y)$ by moving everything to one side of the equation (and the other side can be 0).

Take the partial derivative for x and the partial derivative for y.

$$\frac{dy}{dx} = -\frac{F_x}{F_y}$$

Use the partial derivatives to find the implicit derivative of

$$x^2 - xy + y^2 = \ln(y)$$

$$F(x, y) = x^2 - xy + y^2 - \ln(y)$$

$$F_x = 2x - y$$

$$F_y = -x + 2y - \frac{1}{y}$$

$$\frac{dy}{dx} = -\frac{2x - y}{-x + 2y - \frac{1}{y}}$$

Vs.

$$\frac{2x - y}{\frac{1}{y} + x - 2y} = y'$$

Related Rates

Problems where all the variables in the equation depend on an unstated independent variable (time)

Example.
Suppose the border of a town is roughly circular, and the radius of that circle has been increasing at a rate of 0.1 miles each year. Find how fast the area of the town has been increasing when the radius is 5 miles.

(similar to ripples in a pond)

$$A = \pi r^2$$

$$A(t) = \pi[r(t)]^2$$

When we take the derivative with respect to time, all the function variables are implicit (i.e. apply the chain rule)

$$\frac{dA}{dt} = \pi(2r)\frac{dr}{dt}$$

In these problems they will give you all but one rate and the value of other variables. Solve for the missing one.

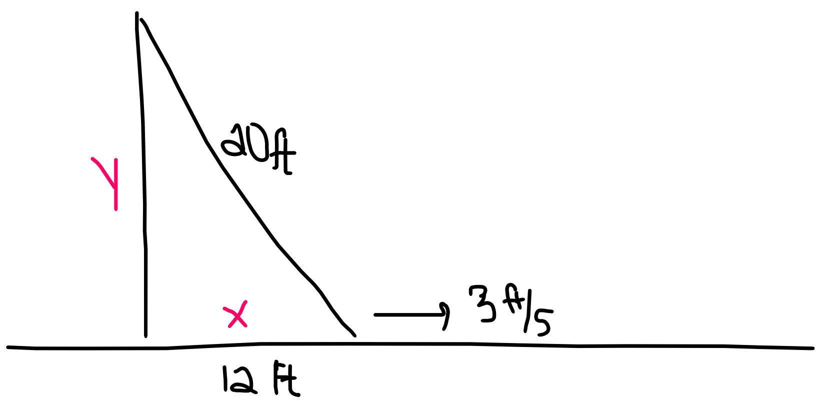
$$\frac{dr}{dt} = 0.1\frac{mi}{yr}$$

$$r = 5$$

$$\frac{dA}{dt} = \pi(2)(5)(0.1) = \pi$$

The area is increasing by $\pi \frac{mi^2}{yr}$

Example.
A young woman and her boyfriend plan to elope, but she must rescue him from his mother who has locked him in his room. The young woman has placed a 20 foot long ladder against his house and is knocking on his window when his mother begins pulling the bottom of the ladder away from the house at a rate of 3 feet per second. How fast is the top of the ladder (and the young couple) falling when the bottom of the ladder is 12 feet from the bottom of the wall?



$$x^2 + y^2 = 20^2$$

$$x^2 + y^2 = 400$$

$$2x\left(\frac{dx}{dt}\right) + 2y\left(\frac{dy}{dt}\right) = 0$$

$$x = 12, \frac{dx}{dt} = 3, y = ?$$

$$12^2 + y^2 = 400$$

$$y^2 = 256, y = 16$$

$$2(12)(3) + 2(16)\left(\frac{dy}{dt}\right) = 0$$

$$72 + 32\frac{dy}{dt} = 0$$

$$32\frac{dy}{dt} = -72$$

$$\frac{dy}{dt} = -\frac{72}{32} = -\frac{9}{4} = -2.25\frac{ft}{s}$$