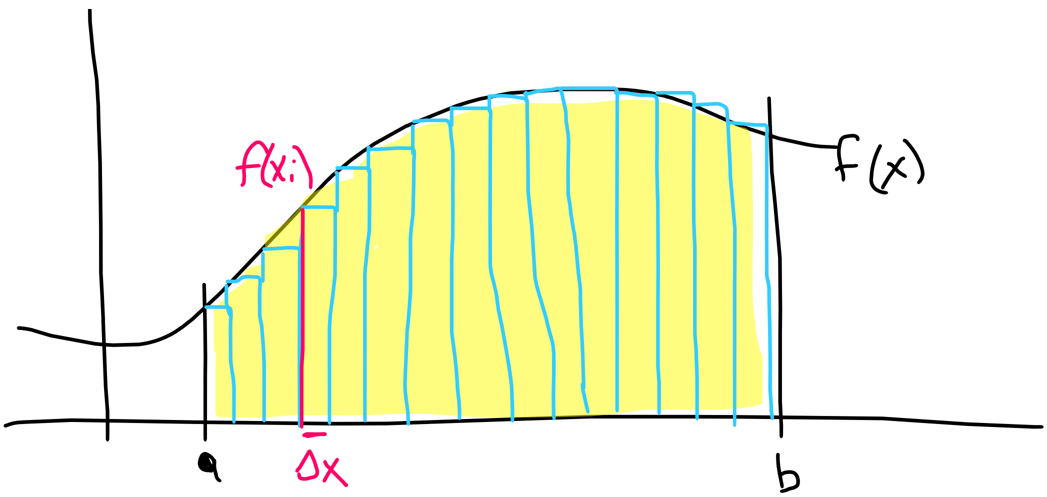
3/27/2024

Area/Definite Integrals (3.1)
Properties of Definite Integrals (3.2)
Fundamental Theorem of Calculus

Area under a curve
Based on estimations using the area of a rectangle:

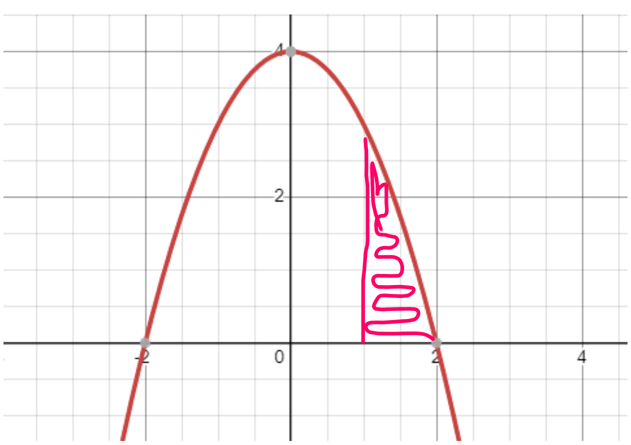
$$A = lw = f(x_i)\Delta x$$



Area estimate under the curve = $\sum_{i=1}^{\infty} f(x_i)\Delta x$

$$\Delta x = \frac{b-a}{n}$$

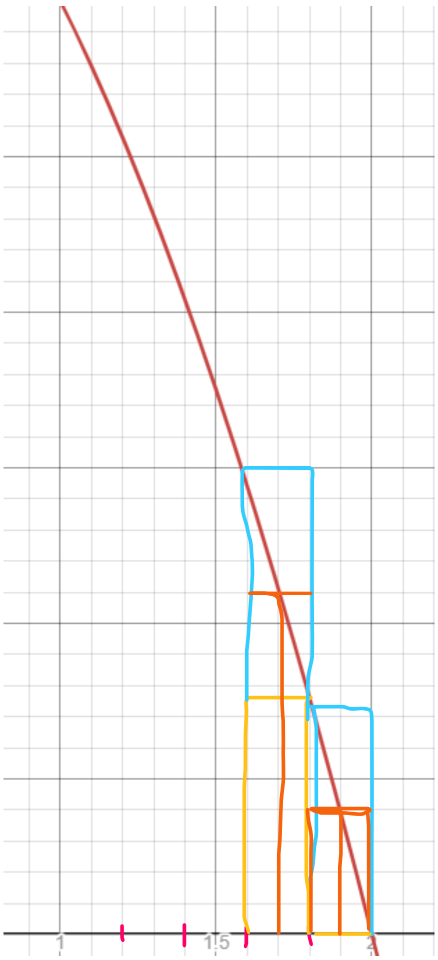
Estimate the area under the curve for the function $f(x) = 4 - x^2$, on the interval [1,2], with n=5 rectangles.



$$\Delta x = \frac{b-a}{n} = \frac{2-1}{5} = \frac{1}{5} = 0.2$$

The potential interval breaks (a partition)

$$\{1, 1.2, 1.4, 1.6, 1.8, 2\} = \{a, a + \Delta x, a + 2\Delta x, \ldots, b\}$$
$$(x_0, x_1, x_2, \ldots., x_n\}$$



For the lefthand rule: we will use the values {1, 1.2, 1.4, 1.6, 1.8} to do our estimations
For the righthand rule: we will use the values {1.2, 1.4, 1.6, 1.8, 2} to do our estimations
For the midpoint rule: we will use the values {1.1, 1.3, 1.5, 1.7, 1.9} to do our estimations

For the lefthand rule:
The first rectangle: $A_1 = f(x_0)\Delta x = f(1)\left(\frac{1}{5}\right) = (4 - 1^2)\left(\frac{1}{5}\right) = 3\left(\frac{1}{5}\right) = \frac{3}{5} = 0.6$
The second rectangle: $A_2 = f(x_1)\Delta x = f(1.2)\left(\frac{1}{5}\right) = (4 - 1.2^2)\left(\frac{1}{5}\right) = (2.56)\left(\frac{1}{5}\right) = 0.512$
The third rectangle: $A_3 = f(x_2)\Delta x = f(1.4)\left(\frac{1}{5}\right) = (4 - 1.4^2)\left(\frac{1}{5}\right) = (2.04)\left(\frac{1}{5}\right) = 0.408$
The fourth rectangle: $A_4 = f(x_3)\Delta x = f(1.6)\left(\frac{1}{5}\right) = (4 - 1.6^2)\left(\frac{1}{5}\right) = (1.44)\left(\frac{1}{5}\right) = 0.288$
The fifth rectangle: $A_5 = f(x_4)\Delta x = f(1.8)\left(\frac{1}{5}\right) = (4 - 1.8^2)\left(\frac{1}{5}\right) = (0.76)\left(\frac{1}{5}\right) = 0.152$

Our estimate from the lefthand rule is to add up our rectangles:

$$\sum_{i=1}^{5} f(x_{i-1})\Delta x = 0.6 + 0.512 + 0.408 + 0.288 + 0.152 = 1.96$$

$$\left(\frac{1}{5}\right)[3 + 2.56 + 2.04 + 1.44 + 0.76] = 1.96$$

The righthand rule:
The first rectangle: $A_1 = f(x_1)\Delta x = f(1.2)\left(\frac{1}{5}\right) = (4 - 1.2^2)\left(\frac{1}{5}\right) = (2.56)\left(\frac{1}{5}\right) = 0.512$
The second rectangle: $A_2 = f(x_2)\Delta x = f(1.4)\left(\frac{1}{5}\right) = (4 - 1.4^2)\left(\frac{1}{5}\right) = (2.04)\left(\frac{1}{5}\right) = 0.408$
The third rectangle: $A_3 = f(x_3)\Delta x = f(1.6)\left(\frac{1}{5}\right) = (4 - 1.6^2)\left(\frac{1}{5}\right) = (1.44)\left(\frac{1}{5}\right) = 0.288$
The fourth rectangle: $A_4 = f(x_4)\Delta x = f(1.8)\left(\frac{1}{5}\right) = (4 - 1.8^2)\left(\frac{1}{5}\right) = (0.76)\left(\frac{1}{5}\right) = 0.152$
The fifth rectangle: $A_5 = f(x_5)\Delta x = f(2)\left(\frac{1}{5}\right) = (4 - 2^2)\left(\frac{1}{5}\right) = 0\left(\frac{1}{5}\right) = 0$

Our estimate from the righthand rule is to add up our rectangles:

$$\sum_{i=1}^{5} f(x_i)\Delta x = 0.512 + 0.408 + 0.288 + 0.152 + 0 = 1.36$$

Given the orientation of the curve (decreasing) the lefthand rule is giving an over-estimate, and the righthand rule is giving an under-estimate.

For the midpoint rule:
The first rectangle: $A_1 = f\left(\frac{x_1+x_0}{2}\right)\Delta x = f(1.1)\left(\frac{1}{5}\right) = (4 - 1.1^2)\left(\frac{1}{5}\right) = (2.79)\left(\frac{1}{5}\right) = 0.558$
The second rectangle: $A_2 = f\left(\frac{x_2+x_1}{2}\right)\Delta x = f(1.3)\left(\frac{1}{5}\right) = (4 - 1.3^2)\left(\frac{1}{5}\right) = (2.31)\left(\frac{1}{5}\right) = 0.462$
The third rectangle: $A_3 = f\left(\frac{x_3+x_2}{2}\right)\Delta x = f(1.5)\left(\frac{1}{5}\right) = (4 - 1.5^2)\left(\frac{1}{5}\right) = (1.75)\left(\frac{1}{5}\right) = 0.35$
The fourth rectangle: $A_4 = f\left(\frac{x_4+x_3}{2}\right)\Delta x = f(1.7)\left(\frac{1}{5}\right) = (4 - 1.7^2)\left(\frac{1}{5}\right) = (1.11)\left(\frac{1}{5}\right) = 0.222$
The fifth rectangle: $A_5 = f\left(\frac{x_5+x_4}{2}\right)\Delta x = f(1.9)\left(\frac{1}{5}\right) = (4 - 1.9^2)\left(\frac{1}{5}\right) = (0.39)\left(\frac{1}{5}\right) = 0.078$

Midpoint rule estimate for the area is:

$$\sum_{i=1}^{5} f\left(\frac{x_i + x_{i-1}}{2}\right)\Delta x = 0.558 + 0.462 + 0.35 + 0.222 + 0.078 = 1.67$$

Exact value is $\frac{5}{3} \approx 1.666666666$ ….

Limit process:

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

$$x_i = a + i\Delta x = a + i\left(\frac{1}{n}\right) = a + \frac{i}{n}$$

$$\Delta x = \frac{b-a}{n} = \frac{2-1}{n} = \frac{1}{n}$$

$$A = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x = \lim_{n\to\infty}\sum_{i=1}^{n} f\left(a + \frac{i}{n}\right)\left(\frac{1}{\text{n}}\right)$$

Our particular function is $f(x) = 4 - x^2$

$$\lim_{n\to\infty}\sum_{i=1}^{n} f\left(a + \frac{i}{n}\right)\left(\frac{1}{\text{n}}\right) = \lim_{n\to\infty}\sum_{i=1}^{n}\left(4 - \left(1 + \frac{i}{n}\right)^2\right)\left(\frac{1}{\text{n}}\right)$$

$$\left(4 - \left(1 + \frac{i}{n}\right)^2\right) = 4 - \left(1 + \frac{i}{n}\right)\left(1 + \frac{i}{n}\right) = 4 - \left(1 + \frac{i}{n} + \frac{i}{n} + \frac{i^2}{n^2}\right) = 3 - \frac{2i}{n} - \frac{i^2}{n^2}$$

$$\lim_{n\to\infty}\sum_{i=1}^{n}\left(4 - \left(1 + \frac{i}{n}\right)^2\right)\left(\frac{1}{\text{n}}\right) = \lim_{n\to\infty}\sum_{i=1}^{n}\left(3 - \frac{2i}{n} - \frac{i^2}{n^2}\right)\left(\frac{1}{\text{n}}\right) =$$

$$\lim_{n\to\infty}\left[\sum_{i=1}^{n}(3)\left(\frac{1}{\text{n}}\right) - \sum_{i=1}^{n}\left(\frac{2i}{n}\right)\left(\frac{1}{\text{n}}\right) - \sum_{i=1}^{n}\left(\frac{i^2}{n^2}\right)\left(\frac{1}{\text{n}}\right)\right] =$$

$$\lim_{n\to\infty}\left[\frac{3}{n}\sum_{i=1}^{n} 1 - \frac{2}{n^2}\sum_{i=1}^{n} i - \frac{1}{n^3}\sum_{i=1}^{n} i^2\right]$$

$$\sum_{i=1}^{n} 1 = n, \sum_{i=1}^{n} k = kn$$

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$$

$$\lim_{n\to\infty}\left[\frac{3}{n}\sum_{i=1}^{n} 1 - \frac{2}{n^2}\sum_{i=1}^{n} i - \frac{1}{n^3}\sum_{i=1}^{n} i^2\right] = \lim_{n\to\infty}\left[\frac{3}{n}(n) - \frac{2}{n^2}\left(\frac{n(n+1)}{2}\right) - \frac{1}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right)\right] =$$

$$\lim_{n\to\infty}\left[\frac{3n}{n}-\frac{2}{n^2}\left(\frac{n^2+n}{2}\right)-\frac{1}{n^3}\left(\frac{2n^3+3n^2+n}{6}\right)\right]=\lim_{n\to\infty}\left[3-\left(\frac{n^2+n}{n^2}\right)-\left(\frac{2n^3+3n^2+n}{6n^3}\right)\right]=$$

$$\lim_{n\to\infty}\left[3-\left(\frac{n^2}{n^2}+\frac{n}{n^2}\right)-\left(\frac{2n^3}{6n^3}+\frac{3n^2}{6n^3}+\frac{n}{6n^3}\right)\right]=\lim_{n\to\infty}\left[3-\left(1+\frac{1}{n}\right)-\left(\frac{1}{3}+\frac{1}{2n}+\frac{1}{6n^2}\right)\right]=3-1-\frac{1}{3}=$$

$$2-\frac{1}{3}=\frac{5}{3}$$

$$n(n+1)(2n+1)=n(2n^2+n+2n+1)=2n^3+3n^2+n$$

The fundamental Theorem of Calculus:
Relates the area under the curve $f(x)$ (between the curve and the x-axis), to a definite integral:

$$\int_a^b f(x)dx=\lim_{n\to\infty}\sum_{i=1}^{n}f(x_i)\Delta x=F(b)-F(a)$$

$$\int_1^2(4-x^2)dx=4x-\frac{x^3}{3}\Big|_1^2=\left[4(2)-\frac{(2)^3}{3}\right]-\left[4(1)-\frac{(1)^3}{3}\right]=\left[8-\frac{8}{3}\right]-\left[4-\frac{1}{3}\right]=\frac{16}{3}-\frac{11}{3}=\frac{5}{3}$$

In TI:
MATH menu →9
fnInt(function,x,lower,upper)

fnInt(4-x^2,x,1,2) = 5/3

Properties of integrals (definite and indefinite)

Any coefficient that does not depend on the variable being integrated can be pulled out of the integral:

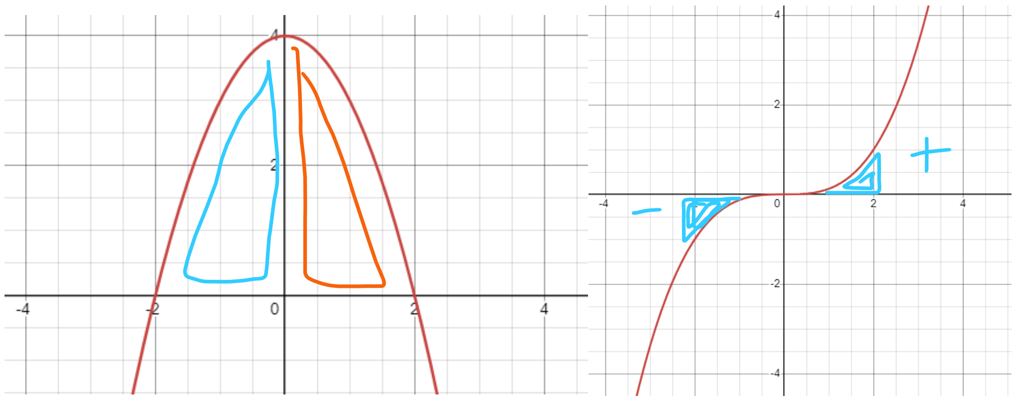
$$\int kf(x)dx=k\int f(x)dx$$

$$\int\big(f(x)\pm g(x)\big)dx=\int f(x)dx\pm\int g(x)dx$$

$$\int_a^b f(x)dx=\int_a^c f(x)dx+\int_c^b f(x)dx$$

$$\int_a^b f(x)dx=-\int_b^a f(x)dx$$

$$\int_{-a}^{a} f(x)dx=\begin{cases}0, if\ f(x)\ is\ odd\\ 2\int_0^a f(x)dx\,, if\ f(x)\ is\ even\end{cases}$$



Left=even, right=odd

If you want the geometric area under the curve (bounded by the curve and y=0/x-axis), then split the integral at the point where the sign changes, and then make the negative side positive (absolute value) before adding to the positive side.

Accumulation function:

$$F(x) = \int_0^x f(t)dt$$

$f(t)$ as a rate function, and the integral is telling you how quickly the total is adding up after the rate is applied for a certain period of time.

Fundamental Theorem Calculus (The second fundamental theorem… or part 2)

$$\frac{d}{dx}[F(x)] = \frac{d}{dx}\left[\int_0^x f(t)dt\right] = f(x)$$

$$\frac{d}{dx}[F(x)] = \frac{d}{dx}\left[\int_0^{g(x)} f(t)dt\right] = f(g(x))g'(x)$$

Example.
Given the accumulation function $A(x) = \int_1^x (e^{2t} - \ln t)dt$, find $A'(x)$.

$$A'(x) = \frac{d}{dx}\left[\int_1^x (e^{2t} - \ln t)dt\right] = e^{2x} - \ln x$$

Example.
Given the accumulation function $B(x) = \int_a^{x^3} e^{t^2} - t\ln(t-1)\, dt$

Find the derivative, $B'(x)$.

$$B'(x) = \frac{d}{dx}\left[\int_a^{x^3} e^{t^2} - t\ln(t-1)\,dt\right] = \left(e^{(x^3)^2} - x^3\ln(x^3-1)\right)(3x^2) = \left(e^{x^6} - x^3\ln(x^3-1)\right)3x^2$$

$$= 3x^2e^{x^6} - 3x^5\ln(x^3-1)$$

Example.

Find the derivative of the function $C(x) = \int_x^{x^2}(\ln t + t)dt$

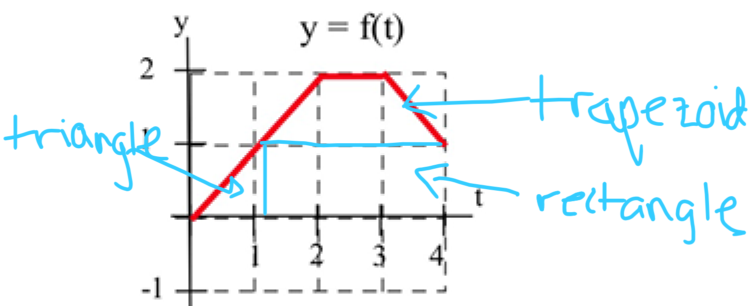
$$C(x) = \int_x^0 (\ln t + t)dt + \int_0^{x^2} (\ln t + t)dt = \int_0^{x^2} (\ln t + t)dt - \int_0^x (\ln t + t)dt$$

$$C'(x) = \frac{d}{dx}\left[\int_0^{x^2} (\ln t + t)dt - \int_0^x (\ln t + t)dt\right] = (\ln x^2 + x^2)(2x) - (\ln x + x)$$

Some of the problems:

If you have to find the area of a region, the general process is to set up a definite integral. Your limits are the interval, and the function inside the integral is the function you are finding the area under.

However, sometimes you can short-cut the process by using basic geometry.



If my goal is to find the area, I can break this up into regular shapes. I don't need to find an exact formula for f(x).

If the problem gives you a rate of growth and then asks for the total (accumulated).

A company determines their marginal cost for production, in dollars per item, is $MC(x) = \frac{4}{\sqrt{x}} + 2$ when producing x thousand items. Find the cost of increasing production from 4 thousand items to 5 thousand items.

$$= \int_4^5 \frac{4}{\sqrt{x}} + 2\,dx$$

$$C(5) - C(4) =$$

You may need to do some arithmetic before doing the antiderivatives:

$$\int_0^1 \frac{(x^5 - x^3)}{x^2} dx = \int_0^1 x^3 - x \, dx$$

Integration techniques: substitution, integration by parts.