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Growth and Decay
Elasticity of Demand (2.10)
Antidifferentiation (3.3)

Growth and Decay

$$P(t) = P_0 e^{kt}$$

P_0 initial population
k is the rate of growth (if positive) or the rate of decay (if negative)
t is the time unit (depends on the rate)

Suppose that a population is growing at a rate of 10% per decade. The initial population in 2010 is 1.5 million. Find the prediction for the population in 2030.

$$P(0) = 1.5\ million = P_0$$
$$k = 0.1$$
$$t = 2$$

$$P(t) = 1.5e^{0.1t}$$

$$P(2) = 1.5e^{0.1\times 2} = 1.83\ million$$

How long will it take for the population to double?

$$3 = 1.5e^{0.1t}$$
$$2 = e^{0.1t}$$
$$\ln 2 = \ln(e^{0.1t}) = 0.1t$$

$$\frac{\ln 2}{0.1} \approx 6.93\ decades, 69.3\ years$$

What is the rate of population growth when the population reaches 3 million? How fast is the population growing per year (or per decade)?

$$P'(t) = 0.15e^{0.1t}$$
$$P'(6.93) \approx 0.3\ per\ decade = 300{,}000\ per\ decade\ or\ 30{,}000\ per\ year$$

Example.
Suppose that a carbon sample from an object has 22% of the carbon-14 of a brand new sample. Use that information to find an estimate for the age of the object. The half-life of carbon-14 is around 5600 years.

$$0.5 = 1.00e^{k(5600)}$$
$$\ln\left(\frac{1}{2}\right) = 5600k$$

$$k = \frac{\ln(0.5)}{5600} \approx -1.24 \times 10^{-4}$$

$$P(t) = e^{-1.24\times10^{-4}t}$$
$$0.22 = e^{-1.24\times10^{-4}t}$$
$$\ln(0.22) = -1.24 \times 10^{-4}t$$

$$t = \frac{\ln(0.22)}{-1.24 \times 10^{-4}} \approx 12{,}233$$

Linear Approximations

We are interested in the behavior of a function $f(x)$ at point x, near another point a which is fixed. We estimate the rate of change of the function using the derivative at the point a, and the linear function passing through (a,f(a)) will be an estimate for the f(x) as long as x remains sufficiently close to a.

$$f(x) \approx f'(a)(x-a) + f(a)$$

$$\Delta f \approx f'(a)\Delta x$$

Example.

Suppose we want to estimate the value of $\sqrt{17}$ using a linear approximation (differentials). What is the function we are estimating with? $f(x) = \sqrt{x}$. What is a perfect square that is close in value to 17?

$a = 16, \Delta x = (x-a) = 17 - 16 = 1$

$$f'(x) = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$$

$$f'(a) = f'(16) = \frac{1}{2\sqrt{16}} = \frac{1}{8}$$

$$\Delta f \approx f'(a)\Delta x = \frac{1}{8}(1)$$

$$f(17) \approx \frac{1}{8}(1) + f(16) = \frac{1}{8} + 4 = 4.125$$

How good is this estimate?

$$f(17) = \sqrt{17} = 4.123 \ldots$$

Only differs in the third decimal place.

Elasticity of Demand

$$E = \left|\frac{p}{q} \times \frac{dq}{dp}\right|$$

Interpret:

E<1 the demand is inelastic, raising prices increases revenue

E>1, the demand is elastic, raising prices decreases revenue
E=1, the demand is called unitary, a critical point of the revenue

If the price increases by 1%, then the demand will decrease by E%

Example.

$$p = 300 - 0.02q$$

Find the elasticity of demand when the price is $70.

In the formula, we have $\frac{dq}{dp}$, now we have p(q). First, we have to solve for q in terms of p.

$$p - 300 = -0.02q$$

$$\frac{p - 300}{-0.02} = q$$

$$-50p + 15000 = q(p)$$

$$\frac{dq}{dp} = -50$$

$$p = 70, q = 11{,}500$$

$$E = \left|\frac{70}{11{,}500} \times -50\right| = \frac{7}{23} \approx 0.304 \ldots$$

The demand is inelastic and we can raise prices to increase revenue.

If we raise prices by 1%, then the demand will only drop by 0.3%

Example.

$$q = 400 - p^2$$

Find E when p=5, and when p=15

$$\frac{dq}{dp} = -2p$$

$$E = \frac{p}{q} \times \left|\frac{dq}{dp}\right|$$

$$q(5) = 400 - 25 = 375$$
$$q(15) = 400 - 225 = 175$$

$$E(5) = \frac{5}{375} \times |-2(5)| = \frac{50}{375} = \frac{2}{15} \approx 0.1333 \ldots$$

$$E(15) = \frac{15}{175} \times |-2(15)| = \frac{450}{175} = \frac{18}{7} \approx 2.57 \ldots$$

When the price is \$5, the demand is inelastic, and increasing the price by 1% will decrease the demand by 0.13%.

When the price is \$15, the demand is elastic, and increasing the price by 1% will decrease demand by 2.57%.

Antiderivatives

In Chapter 2, we learned how to take derivatives, and in Chapter 3, we are going to learn how to do antiderivatives (reverse the process we did in Chapter 2).

The question we are asking is: if the given function is thought of as a derivative, what is the function we started with? What function would we have to have started with in order to obtain this function as its derivative?

Integral notation:

$$\int f(x)dx = take\ the\ antiderivative\ of\ f(x) = F(x)$$

Indefinite integral == generates a function and not a number
Definite integrals will have limits and they give us a number and not a function == $\int_a^b f(x)dx$

Properties of integrals

$$\int kf(x)dx = k\int f(x)dx$$

$$\int f(x) \pm g(x)dx = \int f(x)dx \pm \int g(x)dx$$

Recall our derivative rules… and then reverse them:

Derivative Rule	Antiderivative Rule
$\frac{d}{dx}[x^n] = nx^{n-1}$	$\int x^n dx = \frac{x^{n+1}}{n+1} + C, n \neq -1$
$\frac{d}{dx}[\ln x] = \frac{1}{x}$	$\int \frac{1}{x} dx = \ln\|x\| + C$
$\frac{d}{dx}[e^x] = e^x$	$\int e^x dx = e^x + C$
$\frac{d}{dx}[a^x] = (\ln a)a^x$	$\int a^x dx = \frac{a^x}{\ln a} + C$
$\frac{d}{dx}[\log_a x] = \frac{1}{(\ln a)x}$	$\int \frac{1}{x} dx = \ln(a)\log_a\|x\| + C$
$\frac{d}{dx}[k] = 0$	$\int 0\ dx = C$

$\frac{d}{dx}[x] = 1$	$\int dx = x + C$

$$\frac{df}{dx} = g(x)$$

$$df = g(x)dx$$

$$f(x) = \int df = \int g(x)dx$$

Example.

$$\int 3x^7 - 15\sqrt{x} + \frac{14}{x^2}\,dx = \int \left(3x^7 - 15\sqrt{x} + \frac{14}{x^2}\right)dx =$$

$$\int 3x^7 - 15x^{\frac{1}{2}} + 14x^{-2}\,dx = 3\int x^7\,dx - 15\int x^{\frac{1}{2}}\,dx + 14\int x^{-2}\,dx =$$

$$3\left(\frac{x^8}{8}\right) - 15\left(x^{\frac{3}{2}} \times \frac{2}{3}\right) + 14\left(\frac{x^{-1}}{-1}\right) + C = \frac{3}{8}x^8 - 10\sqrt{x^3} - \frac{14}{x} + C$$

Example.

$$\int e^x + 16 - \frac{12}{x}dx = \int e^x dx + 16\int dx - 12\int \frac{1}{x}dx =$$

$$e^x + 16x - 12\ln|x| + C$$

Suppose you have a function $s(x)$ is the positive of a particle along a curve. We know that the derivative of the positive $s'(x) = v(x)$, the velocity of the particle. And we know that the derivative of the velocity $v'(x) = a(x) = s''(x)$ is the acceleration of the particle at x.

Suppose we are falling in Earth's gravity from an initial height of 300 feet, with an initial velocity of +10 feet/sec, and Earth's gravity is -32 ft/sec^2.

$$a(t) = -32$$

If I want to find the velocity at any point in time, I need to take the antiderivative of the acceleration.

$$v(t) = \int -32\,dt = -32t + C_1$$

$$v(0) = 10$$

$$-32(0) + C_1 = 10$$

$$C_1 = 10$$

$$v(t) = -32t + 10$$

Find the position equation by finding the antiderivative of the velocity.

$$s(t) = \int v(t)dt = \int -32t + 10\,dt = -\frac{32t^2}{2} + 10t + C_2$$

$$s(t) = -16t^2 + 10t + C_2$$
$$s(0) = 300$$

$$-16(0)^2 + 10(0) + C_2 = 300$$
$$C_2 = 300$$

$$s(t) = -16t^2 + 10t + 300$$

Only apply until the ball hits the ground when s(t)=0.

How high is the ball when two seconds have passed?

$$s(2) = -16(2^2) + 10(2) + 300 = 256\,feet$$

How fast is it going? Speed is the absolute value of the velocity.

$$v(2) = -32(2) + 10 = -54$$

It’s at 54 ft/sec downward.

Example.
Find $F(x)$ such that $F'(x) = e^x$, and F(0)=10

$$\int e^x dx = F(x) = e^x + C$$

$$F(0) = 10$$

$$e^{(0)} + C = 10$$
$$1 + C = 10$$
$$C = 9$$

$$F(x) = e^x + 9$$

If you have a rate as an equation, you can find the antiderivative of that rate to obtain an accumulation function: The amount earned in the “account”, the amount accumulated since t=0.

From a previous example:

$$P'(t) = 0.15e^{0.1t}$$

Our rate of change in the population at a given point in time.

What is the total accumulated population since t=0?

$$\int 0.15e^{0.1t}dt = 0.15\frac{e^{0.1t}}{0.1} + C = 1.5e^{0.1t} + C = P(t)$$

$$P(0) = 1.5\, million$$
$$1.5 = 1.5e^{0.1(0)} + C = 1.5(1) + C$$
$$C = 0$$

$$P(t) = 1.5e^{0.1t}$$