2/7/2024

Product Rule, Quotient Rule, Chain Rule
Higher Order Derivatives

Last time we talked about basic rules for simple standalone functions, and their sum and difference. Power rule (polynomial, rational and radical functions), exponential, logarithmic functions.

Product rule

$$(fg)' = f'g + g'f$$

$$h(x) = (x^3 + x)^2 = (x^3 + x)(x^3 + x)$$

Algebra first:

$$h(x) = x^6 + 2x^4 + x^2$$
$$h'(x) = 6x^5 + 8x^3 + 2x$$

$f(x)$	$g(x)$
$x^3 + x$	$x^3 + x$
$f'(x)$	$g'(x)$
$3x^2 + 1$	$3x^2 + 1$

$$f'g' = (3x^2 + 1)(3x^2 + 1) = 9x^4 + 6x^2 + 1$$

Which is not what I got before. So this doesn't work.

$$f'g + g'f = (x^3 + x)(3x^2 + 1) + (3x^2 + 1)(x^3 + x) =$$

$$3x^5 + x^3 + 3x^3 + x + 3x^5 + x^3 + 3x^3 + x = 6x^5 + 8x^3 + 2x$$

That does equal what we got when we FOILed first. So this is the correct formula.

Example.
Find the derivative of $h(x) = \left(x^4 - \sqrt{x}\right)\left(x^3 + 2x - \frac{1}{x}\right)$

$f(x) = x^4 - \sqrt{x} = x^4 - x^{\frac{1}{2}}$	$g(x) = x^3 + 2x - \dfrac{1}{x} = x^3 + 2x - x^{-1}$
$f'(x) = 4x^3 - \dfrac{1}{2}x^{-\frac{1}{2}} = 4x^3 - \dfrac{1}{2\sqrt{x}}$	$g'(x) = 3x^2 + 2 + x^{-2} = 3x^2 + 2 + \dfrac{1}{x^2}$

$$h'(x) = \left(4x^3 - \frac{1}{2\sqrt{x}}\right)\left(x^3 + 2x - \frac{1}{x}\right) + \left(3x^2 + 2 + \frac{1}{x^2}\right)\left(x^4 - \sqrt{x}\right)$$

Example.
Find the derivative of $h(x) = x^2 e^x$

$f(x) = x^2$	$g(x) = e^x$
$f'(x) = 2x$	$g'(x) = e^x$

$$h'(x) = x^2e^x + 2xe^x$$

Example.
Find the derivative of $h(x) = 3^x \ln x$

$f(x) = 3^x$	$g(x) = \ln x$
$f'(x) = (\ln 3)3^x$	$g'(x) = \frac{1}{x}$

$$h'(x) = 3^x\left(\frac{1}{x}\right) + (\ln 3)3^x \ln x$$

Quotient Rule

$$\left(\frac{f}{g}\right)' = \frac{f'g - g'f}{g^2}$$

Example.
Find the derivative of $h(x) = \frac{2x+1}{x-3}$

$f(x) = 2x + 1$	$g(x) = x - 3$
$f'(x) = 2$	$g'(x) = 1$

$$h'(x) = \frac{2(x-3) - (1)(2x+1)}{(x-3)^2} = \frac{2x - 6 - 2x - 1}{(x-3)^2} = \frac{-7}{(x-3)^2}$$

Example.
Find the derivative of $h(x) = \frac{x^5+x^2}{x^3+x-1}$

$f(x) = x^5 + x^2$	$g(x) = x^3 + x - 1$
$f'(x) = 5x^4 + 2x$	$g'(x) = 3x^2 + 1$

$$h'(x) = \frac{(5x^4 + 2x)(x^3 + x - 1) - (3x^2 + 1)(x^5 + x^2)}{(x^3 + x - 1)^2}$$

Example.
Find the derivative of $h(x) = \frac{e^x}{\ln x}$

$f(x) = e^x$	$g(x) = \ln x$
$f'(x) = e^x$	$g'(x) = \frac{1}{x}$

$$h'(x) = \frac{\left(e^x \ln x - \frac{e^x}{x}\right)}{(\ln x)^2} = \frac{e^x \ln x - \frac{e^x}{x}}{\ln^2 x}$$

Example.
Find the derivative of $h(x) = \frac{x^2 e^x}{x^3 - 3}$

$f(x) = x^2 e^x$		$g(x) = x^3 - 3$
$F(x) = x^2$	$G(x) = e^x$	$g'(x) = 3x^2$
$F'(x) = 2x$	$G'(x) = e^x$	
$f'(x) = x^2 e^x + 2xe^x$		

$$h'(x) = \frac{(x^2 e^x + 2xe^x)(x^3 - 3) - (3x^2)(x^2 e^x)}{(x^3 - 3)^2}$$

Chain Rule.
The chain rule is for doing derivatives of composite functions: $f\big(g(x)\big) = (f \circ g)(x)$

Example. $h(x) = (x^4 - x)^7$

$$f(x) = x^7, g(x) = x^4 - x$$

Example. $h(x) = e^{\sqrt{x}}$

Example. $h(x) = \ln(e^x + 1)$

Example. $h(x) = e^{2x}$

Chain rule: $h(x) = f\big(g(x)\big), h'(x) = f'\big(g(x)\big)g'(x)$

Find the derivative of $h(x) = (x^4 - x)^7 \approx (something)^7$

$$h'(x) = 7(something)^6(something)' = 7(x^4 - x)^6(4x^3 - 1)$$

Find the derivative of $h(x) = (x^3 + x)^2$

$$h'(x) = 2(something)^1(something)' = 2(x^3 + x)(3x^2 + 1) = 2(3x^5 + x^3 + 3x^3 + x)$$
$$= 6x^5 + 8x^3 + 2x$$

Example. Find the derivative of $h(x) = \ln(e^x + 1)$

$$h'(x) = \frac{1}{stuff}(stuff)' = \frac{1}{e^x + 1}(e^x) = \frac{e^x}{e^x + 1}$$

Example. Find the derivative of $h(x) = e^{2x}$

$$h'(x) = e^{2x}(2x)' = 2e^{2x}$$

Example. Find the derivative of $h(x) = e^{\sqrt{x}}$

$$h'(x) = e^{\sqrt{x}}\left(\sqrt{x}\right)' = e^{\sqrt{x}}\left(\frac{1}{2}x^{-\frac{1}{2}}\right) = \frac{e^{\sqrt{x}}}{2\sqrt{x}}$$

Example. Find the derivative of $h(x) = \sqrt[3]{e^x - x} = (e^x - x)^{\frac{1}{3}}$

$$h'(x) = \frac{1}{3}(e^x - x)^{-\frac{2}{3}}(e^x - 1) = \frac{e^x - 1}{3\sqrt[3]{(e^x - x)^2}}$$

Example. Find the derivative of $h(x) = \frac{2}{(x^2+1)^5} = 2(x^2 + 1)^{-5}$

$$h'(x) = 2(-5)(x^2 + 1)^{-6}(2x) = -\frac{20x}{(x^2 + 1)^6}$$

$$\left(\frac{f}{g}\right)' = (f\,g^{-1})' = f'(g^{-1}) + (g^{-1})'f = f'g^{-1} - g^{-2}g'f$$

$$= \frac{f'}{g} - \frac{g'f}{g^2} = \frac{f'g}{g^2} - \frac{g'f}{g^2} = \frac{f'g - g'f}{g^2}$$

Combinations of product/quotient and chain rules

Example. Find the derivative of $h(x) = e^{x^2}(x^4 - x^3)^8$

$f(x) = e^{x^2}$	$g(x) = (x^4 - x^3)^8$
$f'(x) = e^{x^2}(x^2)' = 2xe^{x^2}$	$g'(x) = 8(x^4 - x^3)^7(4x^3 - 3x^2)$

$$h'(x) = \left(2xe^{x^2}\right)(x^4 - x^3)^8 + 8e^{x^2}(x^4 - x^3)^7(4x^3 - 3x^2)$$

Example. Find the derivative of $h(x) = \frac{\ln(x^2+10)}{x^7+1}$

$f(x) = \ln(x^2 + 10)$	$g(x) = x^7 + 1$
$f'(x) = \frac{1}{x^2 + 10}(x^2 + 10)' = \frac{2x}{x^2 + 10}$	$g'(x) = 7x^6$

$$h'(x) = \frac{\left((x^7 + 1)\left(\frac{2x}{x^2 + 10}\right) - (7x^6)\ln(x^2 + 10)\right)}{(x^7 + 1)^2}$$

Example. Find the derivative of $h(x) = \sqrt{\frac{5x-6}{3x+4}} = \left(\frac{5x-6}{3x+4}\right)^{\frac{1}{2}}$

$$h'(x) = \frac{1}{2}\left(\frac{5x-6}{3x+4}\right)^{-\frac{1}{2}}\left(\frac{5x-6}{3x+4}\right)' = \frac{1}{2}\left(\frac{3x+4}{5x-6}\right)^{\frac{1}{2}}\left(\frac{5x-6}{3x+4}\right)'$$

$f(x) = 5x - 6$	$g(x) = 3x + 4$
$f'(x) = 5$	$g'(x) = 3$

$$h'(x) = \frac{1}{2}\left(\frac{3x+4}{5x-6}\right)^{\frac{1}{2}}\left[\frac{5(3x+4) - 3(5x-6)}{(3x+4)^2}\right]$$

Example. Find the derivative of $h(x) = \sqrt{\frac{5x-6}{3x+4}} = \left(\frac{5x-6}{3x+4}\right)^{\frac{1}{2}} = \frac{(5x-6)^{\frac{1}{2}}}{(3x+4)^{\frac{1}{2}}} = (5x-6)^{\frac{1}{2}}(3x+4)^{-\frac{1}{2}}$

Multiple chains.

$$F(x) = f\left(g\big(h(x)\big)\right)$$

Find the derivative of $F(x) = \left(e^{\frac{1}{x}} - \ln(x^3 + e^{5x})\right)^{10}$

$$F(x) = (something)^{10}$$

$$F'(x) = 10(something)^9(something)' = 10\left(e^{\frac{1}{x}} - \ln(x^3 + e^{5x})\right)^9\left(e^{\frac{1}{x}} - \ln(x^3 + e^{5x})\right)'$$

$$G(x) = e^{\frac{1}{x}} - \ln(x^3 + e^{5x})$$

$$G'(x) = e^{\frac{1}{x}}\left(\frac{1}{x}\right)' - \frac{1}{x^3 + e^{5x}}(x^3 + e^{5x})' = e^{\frac{1}{x}}\left(-\frac{1}{x^2}\right) - \frac{1}{x^3 + e^{5x}}(3x^2 + 5e^{5x})$$

$$\left(\frac{1}{x}\right)' = (x^{-1})' = -1x^{-2} = -\frac{1}{x^2}$$

$$(x^3 + e^{5x})' = 3x^2 + 5e^{5x}$$

$$F'(x) = 10\left(e^{\frac{1}{x}} - \ln(x^3 + e^{5x})\right)^9\left[e^{\frac{1}{x}}\left(-\frac{1}{x^2}\right) - \frac{1}{x^3 + e^{5x}}(3x^2 + 5e^{5x})\right]$$

Example.

Find the derivative of $F(x) = \ln^4(x^3 - e^{1.1x}) = [\ln(x^3 - e^{1.1x})]^4$

$$F'(x) = 4[\ln(x^3 - e^{1.1x})]^3(\ln(x^3 - e^{1.1x}))' = 4[\ln(x^3 - e^{1.1x})]^3\left(\frac{1}{x^3 - e^{1.1x}}(x^3 - e^{1.1x})'\right) =$$

$$4[\ln(x^3 - e^{1.1x})]^3 \left(\frac{1}{x^3 - e^{1.1x}}(3x^2 - 1.1e^{1.1x})\right)$$

Higher Order Derivatives.
Derivatives of derivatives…

First derivative – take the derivative one time, $f'(x), y', \frac{df}{dx}$
Second derivative – take the derivative twice, take the derivative of the derivative, $f''(x), y'', \frac{d^2f}{dx^2}$
Third derivatives – take the derivatives of the second derivative, $f'''(x), y''', \frac{d^3f}{dx^3}$
Fourth derivative – take the derivative of the third derivative, $f^{IV}(x), f^{(4)}(x), \frac{d^4f}{dx^4}$

Nth derivative – take the derivative n times, $f^{(n)}(x)$

Example. Find the fifth derivative of $f(x) = x^8 - x^3 + x - x^{\frac{1}{3}} + 4x^{-1}$

$$f'(x) = 8x^7 - 3x^2 + 1 - \frac{1}{3}x^{-\frac{2}{3}} - 4x^{-2}$$
$$f''(x) = 56x^6 - 6x + \frac{2}{9}x^{-\frac{5}{3}} + 8x^{-3}$$
$$f'''(x) = 336x^5 - 6 - \frac{10}{27}x^{-\frac{8}{3}} - 24x^{-4}$$
$$f^{IV}(x) = 1680x^4 + \frac{80}{81}x^{-\frac{11}{3}} + 96x^{-5}$$

$$f^{V}(x) = 6720x^3 - \frac{880}{243}x^{-\frac{14}{3}} - 480x^{-6}$$

Position for the original, velocity is the derivative, acceleration is the second derivative

The first derivative being positive indicates that the original function is increasing, while the first derivative being negative indicates that the original function is decreasing.

The second being positive means that the graph is bowl-shaped facing upward. The second derivative being negative means the graph is bowl-shaped facing downward.

Maxima and minima can occur when the derivative is zero. Inflection points when the second derivative is 0.