2/21/2024

Multivariable Optimization (4.3)
Curve Sketching (2.8)
Review for Exam #1

Multivariable Optimization in two variables (only)
In the one-variable case, we had both a first derivative test and a second derivative test.
The first derivative in two dimensions involves vectors (which we are not required to know for this course), so we won't do that. There is a second derivative test, second partial test, which we can do.

Second partials test is also sometimes referred to as the D-test.

$$D = f_{xx}(x_0, y_0) f_{yy}(x_0, y_0) - \left(f_{xy}(x_0, y_0)\right)^2$$

Possible outcomes:
4 things can happen:
1) Maximum: D is positive, f_{xx} is negative then that means it is concave down and a maximum (f_{yy} will match)
2) Minimum: D is positive, f_{xx} is positive then that means it is concave up and a minimum (f_{yy} will match)
3) Saddle Point: D is negative
4) Cannot be determined: $D = 0$

Example.
Find the critical point(s) and classify them for the function $f(x, y) = 2xy - x^2 - 2y^2 + 6x + 4$.

$$f_x = 2y - 2x + 6$$
$$f_y = 2x - 4y$$

$$f_{xx} = -2$$
$$f_{yy} = -4$$
$$f_{xy} = 2$$

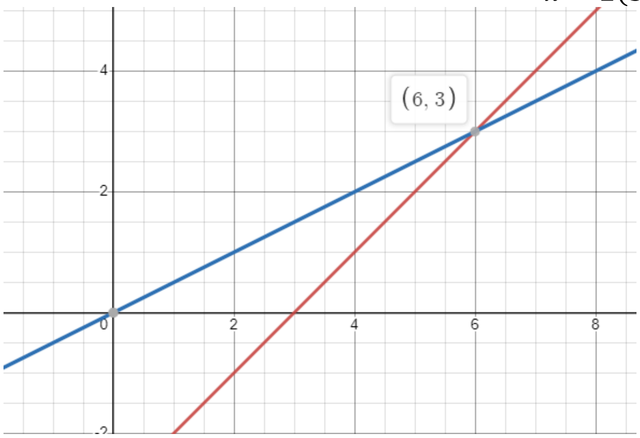
$$D = (-2)(-4) - (2)^2 = 8 - 4 = 4$$
D is positive, and f_{xx} is negative, concave downs means it's a maximum.

Where is the critical point? Occurs when all the partial derivatives are 0 at the same time. Set both of our first partials equal to zero and then solve them as a system.

$$2y - 2x + 6 = 0$$
$$2x - 4y = 0$$
$$2x = 4y$$
$$x = 2y$$

$$2y - 2(2y) + 6 = 0$$
$$2y - 4y = -6$$

$$-2y = -6$$
$$y = 3$$
$$x = 2(3) = 6$$



Critical point is at $(6,3)$ and it is a maximum.

Example.
Find all the critical points and characterize them for the function $f(x,y) = x^3 + y^3 + 3x^2 - 3y^2 - 8$

$$f_x = 3x^2 + 6x = 0$$
$$f_y = 3y^2 - 6y = 0$$

$$f_{xx} = 6x + 6$$
$$f_{yy} = 6y - 6$$
$$f_{xy} = 0$$

$$3x^2 + 6x = 3x(x+2) = 0$$
$$x = 0, x = -2$$

$$3y^2 - 6y = 3y(y-2) = 0$$
$$y = 0, y = 2$$

4 critical points: $(0,0), (0,2), (-2,0), (-2,2)$

D-test for (0,0): $D = [6(0) + 6][6(0) - 6] - 0^2 = 6(-6) = -36$ A saddle point
D-test for (0,2): $D = [6(0) + 6][6(2) - 6] - 0^2 = 6(6) = 36$ either a maximum or a minimum; concave up means minimum
D-test for (-2,0): $= [6(-2) + 6][6(0) - 6] - 0^2 = (-6)(-6) = 36$ either a maximum or a minimum; concave down, so it's a maximum
D-test for (-2,2): $[6(-2) + 6][6(2) - 6] - 0^2 = (-6)(6) = -36$ A saddle point.

Example.
Find all the critical points and characterize them for the function $f(x,y) = 9x^3 + \frac{1}{3}y^3 - 4xy$

$$f_x = 27x^2 - 4y$$

$$f_y = y^2 - 4x$$

$$f_{xx} = 54x$$
$$f_{yy} = 2y$$
$$f_{xy} = 4$$

Critical points.

$$27x^2 - 4y = 0$$
$$y^2 - 4x = 0$$
$$y^2 = 4x$$
$$x = \frac{1}{4}y^2$$

$$27x^2 - 4y = 0$$
$$27\left(\frac{1}{4}y^2\right)^2 - 4y = 0$$

$$\frac{27}{16}y^4 - 4y = 0$$

$$27y^4 - 64y = 0$$
$$y(27y^3 - 64) = 0$$
$$y = 0, 27y^3 = 64$$

$$y = 0, y = \sqrt[3]{\frac{64}{27}} = \frac{4}{3}$$

$$x = \frac{1}{4}y^2$$
$$x = \frac{1}{4}(0)^2 = 0$$

$$x = \frac{1}{4}\left(\frac{4}{3}\right)^2 = \frac{1}{4}\left(\frac{16}{9}\right) = \frac{4}{9}$$

Critical point (0,0) and $\left(\frac{4}{9}, \frac{4}{3}\right)$

D-test for (0,0): $D = 54(0)2(0) - 4^2 = 0 - 16 = -16$ A saddle point.
D-test for $\left(\frac{4}{9}, \frac{4}{3}\right)$: $D = 54\left(\frac{4}{9}\right)2\left(\frac{4}{3}\right) - 4^2 = 24\left(\frac{8}{3}\right) - 16 = 64 - 16 = 48$ either a maximum or a minimum, and the unmixed second partials are both positive, concave up which means a minimum.

Example.
Find all the critical points and characterize them for the function $f(x, y) = xy + 2x - \ln(x^2y), x > 0, y > 0$

$$f(x, y) = xy + 2x - (\ln x^2 + \ln y) = xy + 2x - 2\ln x - \ln y$$

$$f_x = y + 2 - \frac{1}{x}$$
$$f_y = x - \frac{1}{y}$$

$$f_{xx} = \frac{1}{x^2}$$
$$f_{yy} = \frac{1}{y^2}$$
$$f_{xy} = 1$$

$$y + 2 - \frac{1}{x} = 0$$

$$x - \frac{1}{y} = 0$$
$$x = \frac{1}{y}$$

$$y + 2 - \frac{1}{\frac{1}{y}} = y + 2 - y = 0$$

$$2 = 0$$

Not possible.
No critical points on this function.

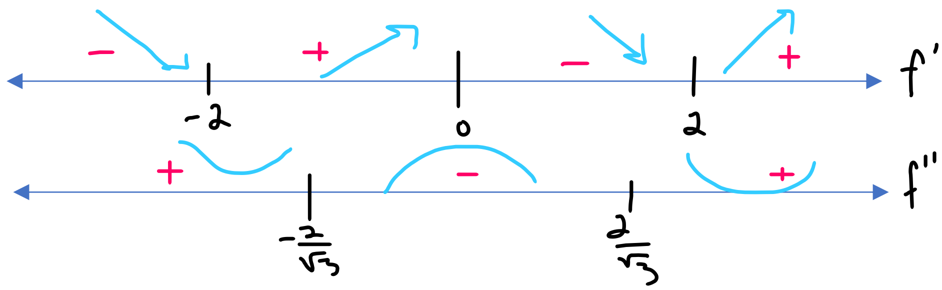
Curve Sketching
Use properties of the derivative, and second derivative (and some algebra) to sketch a graph of a function without technology.

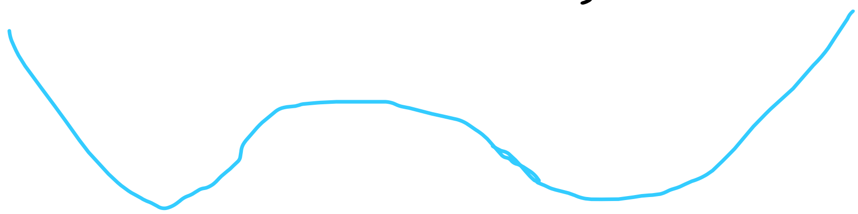
Examples.
Sketch the graph of $f(x) = x^4 - 8x^2 + 3$
$$f'(x) = 4x^3 - 16x$$
$$4x^3 - 16x = 0$$
$$4x(x^2 - 4) = 0$$
$$4x(x-2)(x+2) = 0$$
$$x = 0, 2, -2$$

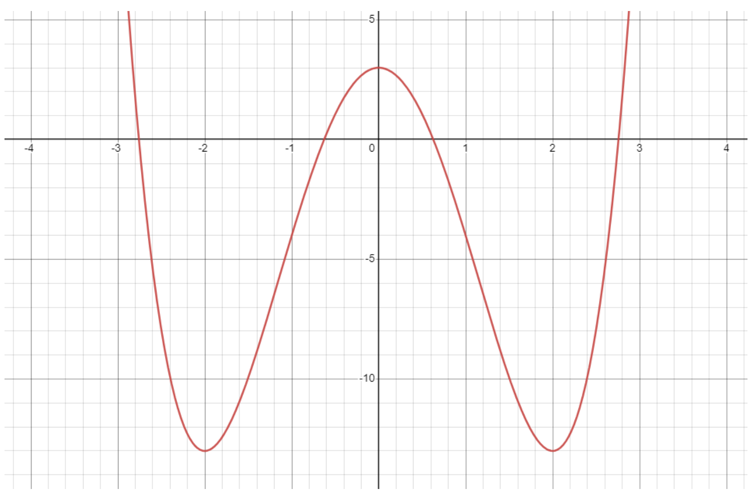
Three critical points.

$$f''(x) = 12x^2 - 16$$
$$12x^2 - 16 = 0$$
$$3x^2 - 4 = 0$$
$$x^2 = \frac{4}{3}$$
$$x = \pm\frac{2}{\sqrt{3}} \approx \pm 1.1547 \ldots$$

Two inflection points.







Example.
Sketch the graph of the function $f(x) = \frac{2}{x^2+1}$

For rational functions, use any horizontal and vertical asymptotes you can find algebraically.

This function has no vertical asymptotes because the denominator cannot be zero. There is a horizontal asymptote at y=0.

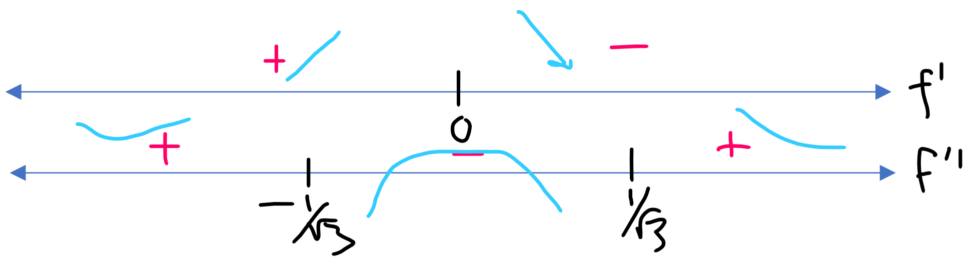
$$f(x) = 2(x^2+1)^{-1}$$
$$f'(x) = 2(-1)(x^2+1)^{-2}(2x) = -4x(x^2+1)^{-2} = \frac{-4x}{(x^2+1)^2}$$

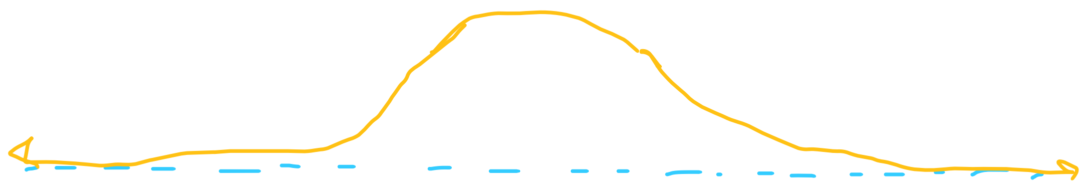
$$-\frac{4x}{(x^2+1)^2} = 0$$
$$-4x = 0$$

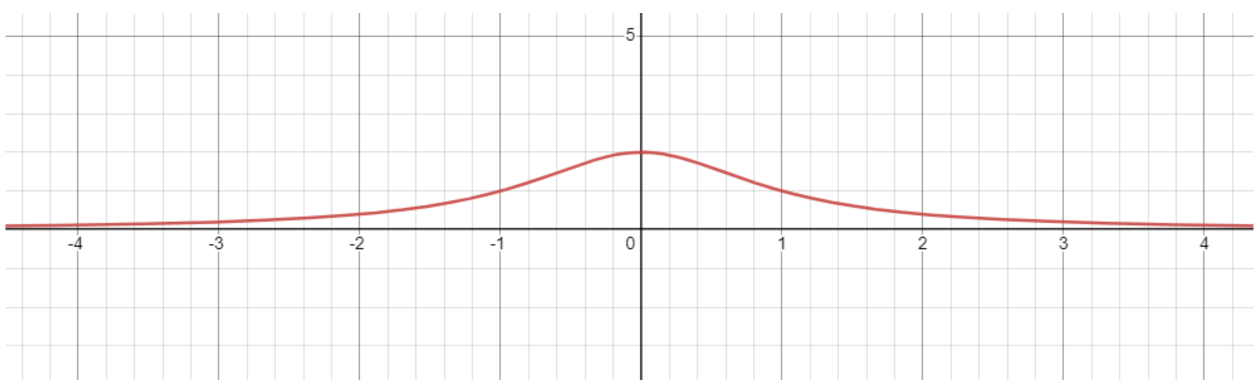
$$x = 0$$

$$f''(x) = -4(x^2+1)^{-2} - 4x(-2)(x^2+1)^{-3}(2x) = -\frac{4}{(x^2+1)^2} + \frac{16x^2}{(x^2+1)^3} = \frac{-4(x^2+1)+16x^2}{(x^2+1)^3}$$

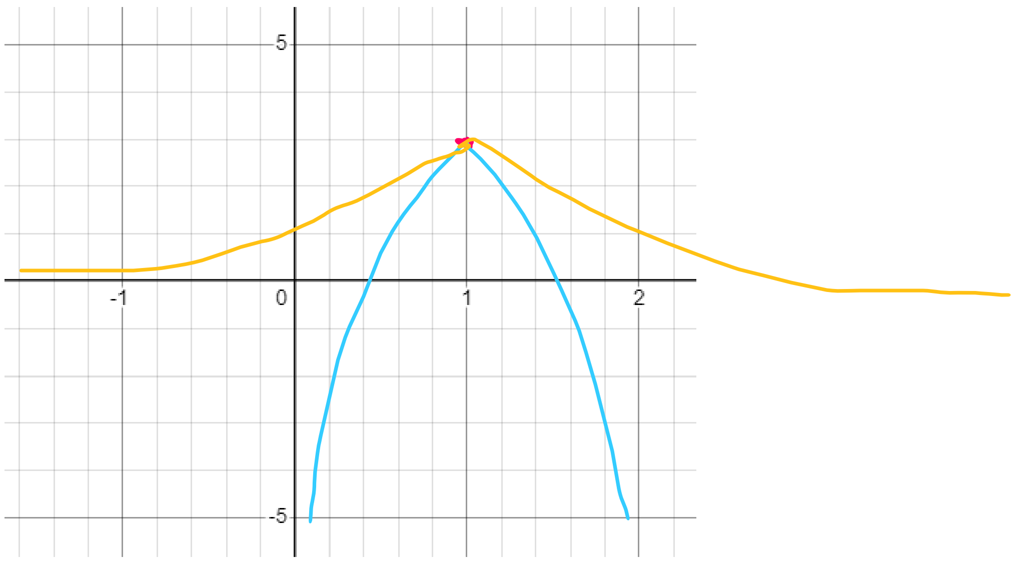
$$-4(x^2+1)+16x^2 = 0$$
$$-4x^2 - 4 + 16x^2 = 0$$
$$12x^2 - 4 = 0$$
$$3x^2 - 1 = 0$$
$$x = \pm\sqrt{\frac{1}{3}} = \pm\frac{1}{\sqrt{3}} \approx 0.577\ldots$$



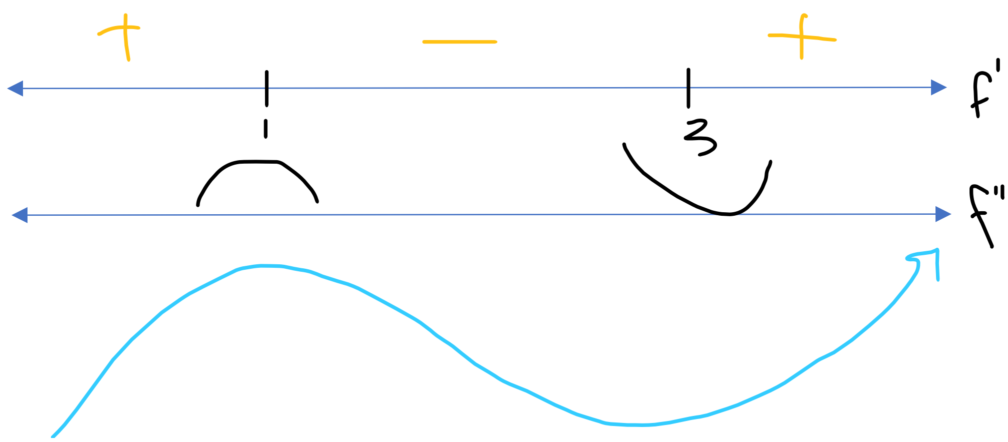




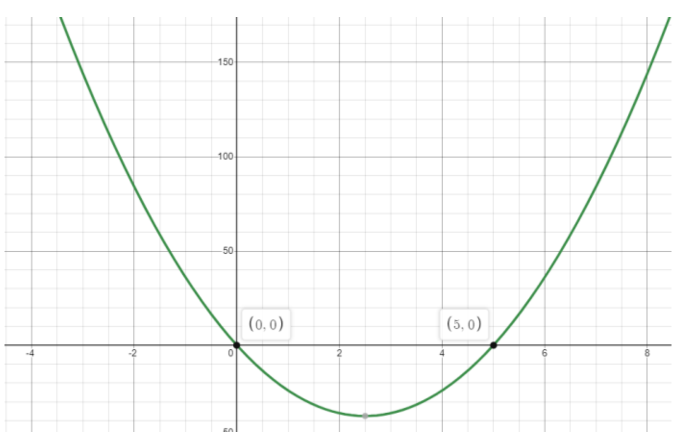
Sketching without a function.
Example.
$f(1) = 3, f'(1) = 0, (1,3)$ is a maximum



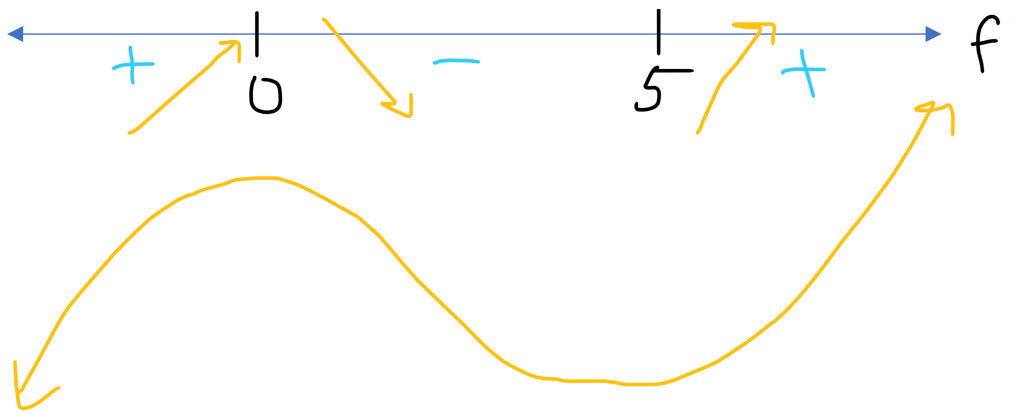
Example.
$f'(x) > 0$ on $(-\infty, 1)$, and $f'(x) < 0$ on $(1,3)$, and $f'(x) > 0$ on $(3, \infty)$
$f''(1) < 0, f''(3) > 0$

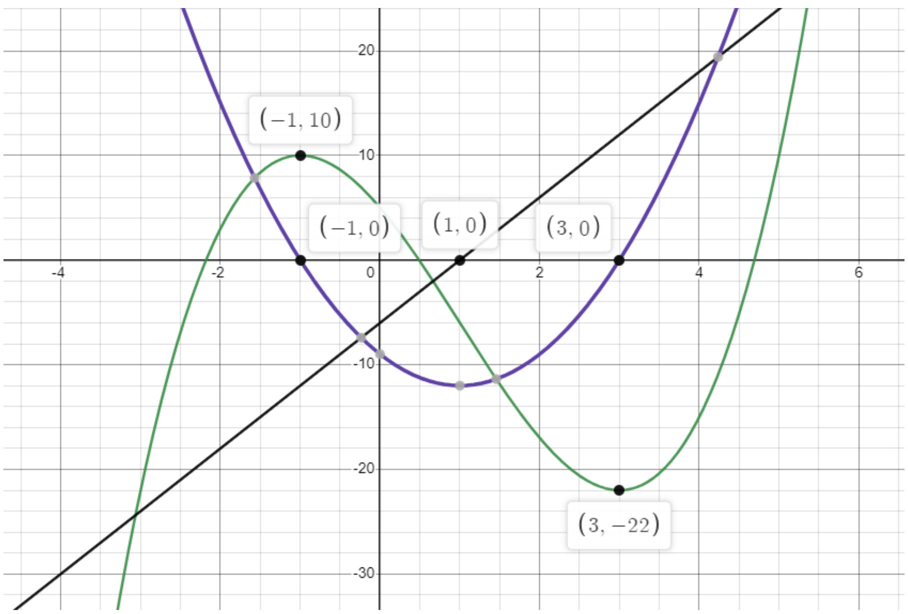


Sketch the original function from the graph of its derivative.



The derivative is equal to 0 at 0 and 5.





F is cubic, the derivative is quadratic, and the second derivative is linear.

Exam #1.
Chapter 2 (up to 2.8), and 4.1, 4.2 of Chapter 4