2/14/2024

Partial Derivatives (4.2)
Optimization (2.7)

$$(3^x)' = \left[\left(e^{\ln 3}\right)^x\right]' = \left(e^{x\ln 3}\right)' = e^{x\ln 3}(\ln 3) = (3^x)\ln 3$$

Product Rule + Chain Rule
Example.

$$h(x) = e^{4x}(x^2+1)^3$$

$f(x) = e^{4x}$	$g(x) = (x^2+1)^3$
$f'(x) = 4e^{4x}$	$g'(x) = 3(x^2+1)^2(2x)$

$$h'(x) = e^{4x}(6x)(x^2+1)^2 + 4e^{4x}(x^2+1)^3$$

Partial Derivatives
It's a derivative of a multivariable function (2 or more variables) taking the derivative of one variable at a time.

Function, $f(x,y)$, the derivative with respect to x, $f_x(x,y) = \frac{\partial f}{\partial x}$; the derivative with respect to y, $f_y(x,y) = \frac{\partial f}{\partial y}$.

$$f_x = \lim_{h\to 0}\frac{f(x+h,y)-f(x,y)}{h}$$

$$f_y = \lim_{h\to 0}\frac{f(x,y+h)-f(x,y)}{h}$$

Example. Find f_x for $f(x,y) = x^2 + 3xy - y^2$

$$\lim_{h\to 0}\frac{f(x+h,y)-f(x,y)}{h} = \lim_{h\to 0}\frac{(x+h)^2 + 3(x+h)y - y^2 - (x^2+3xy-y^2)}{h} =$$

$$\lim_{h\to 0}\frac{x^2 + 2xh + h^2 + 3xy + 3hy - y^2 - x^2 - 3xy + y^2}{h} =$$

$$\lim_{h\to 0}\frac{2xh + h^2 + 3hy}{h} = \lim_{h\to 0}\frac{h(2x+h+3y)}{h} = \lim_{h\to 0}(2x+h+3y) = 2x+3y$$

Essentially, what we do procedurally is that we treat any variable in the expression that we are not using for the derivative as if it is constant, and then take the derivative only of the variable of interest.

$$f_y = 0 + 3x(1) - 2y = 3x - 2y$$

Example.

Find all the first partial derivatives of the function $f(x,y,z) = x^3 - 4xy^2 + 9xyz - y^2z^3 + z^4$

$$f_x = 3x^2 - 4y^2(1) + 9yz(1) - 0 + 0 = 3x^2 - 4y^2 + 9yz$$

$$f_y = 0 - 4x(2y) + 9xz(1) - z^3(2y) + 0 = -8xy + 9xz - 2yz^3$$

$$f_z = 0 - 0 + 9xy(1) - y^2(3z^2) + 4z^3 = 9xy - 3y^2z^2 + 4z^3$$

When do the product and quotient rules apply?

$$ye^y \to yes$$
$$xe^y \to no$$
$$\frac{x+y}{x-y} \to yes$$
$$\frac{e^x}{y} \to no$$

Example.

Find all the first partial derivatives of $f(x,y) = xe^{xy} + y\ln(x+y) - \frac{e^x}{y}$

x	e^{xy}
1	$e^{xy}(y) = ye^{xy}$

$$f_x = xye^{xy} + e^{xy} + y\frac{1}{x+y}(1+0) - \frac{1}{y}e^x = xye^{xy} + e^{xy} + \frac{y}{x+y} - \frac{e^x}{y}$$

y	$\ln(x+y)$
1	$\frac{1}{x+y}(0+1)$

$$f_y = x(e^{xy})(x) + \frac{y}{x+y} + (1)\ln(x+y) - e^x(-1y^{-2}) = x^2e^{xy} + \frac{y}{x+y} + \ln(x+y) + \frac{e^x}{y^2}$$

Example.

Find all the partial derivatives of $f(x,y) = (x^3 - y^2)^5$

$$f_x = 5(x^3 - y^2)^4\left[\frac{\partial(x^3 - y^2)}{\partial x}\right] = 5(x^3 - y^2)(3x^2 - 0) = 15x^2(x^3 - y^2)^4$$

$$f_y = 5(x^3 - y^2)^4\left[\frac{\partial(x^3 - y^2)}{\partial y}\right] = 5(x^3 - y^2)(0 - 2y) = -10y(x^3 - y^2)^4$$

Example.

Find all the partial derivatives of the function $f(x,y,z,w) = \ln(w) + xe^z + y^2w^3 - 4^{xy} + z\ln(z-w)$

$$f_x = 0 + e^z(1) + 0 - 4^{xy}(\ln 4)(y) + 0 = e^z - (\ln 4)y4^{xy}$$

$$f_y = 0 + 0 + w^3(2y) - 4^{xy}(\ln 4)(x) + 0 = 2yw^3 - (\ln 4)x4^{xy}$$

$$f_w = \frac{1}{w} + 0 + y^2(3w^2) + 0 + z\left(\frac{1}{z-w}\right)(0-1) = \frac{1}{w} + 3y^2w^2 - \frac{z}{z-w}$$

$$f_z = 0 + xe^z + 0 + 0 + (1)\ln(z-w) + z\left(\frac{1}{z-w}\right)(1-0) = xe^z + \ln(z-w) + \frac{z}{z-w}$$

Could be asked to evaluate these partial derivatives at particular points.

$$(4^x)' = 4^x(\ln 4)$$
$$(4^{2x})' = 4^{2x}(\ln 4)(2x)' = 4^{2x}(\ln 4)(2)$$

$$(4^{xy})' = 4^{xy}(\ln 4)(xy)'$$

$$f_x(1,2,3,4) = e^3 - (\ln 4)(2)4^{1\times 2} = e^3 - 32\ln 4$$

Higher order partial derivatives
f_{xx} is the second derivative with respect to x… take the derivative with respect to x, and then take the derivative with respect to x again.
f_{yy} is the second derivative with respect to y… take the derivative with respect to y and then take the derivative with respect to y again.
f_{xy} is also a second derivative, but it’s called a mixed partial because the first time we take the derivative it’s for x, and the second time it’s for y.
f_{yx} is the mixed partial, it’s the derivative for y first, and then the derivative for x.

$$f_{xy} = f_{yx}$$

In the three variable case:

$$f_{xy} = f_{yx}$$
$$f_{xz} = f_{zx}$$
$$f_{yz} = f_{zy}$$

In the third derivative case:

$$f_{xyz} = f_{xzy} = f_{yxz} = f_{yzx} = f_{zyx} = f_{zxy}$$
$$f_{xxy} = f_{xyx} = f_{yxx}$$

Consider the function $f(x,y) = 2x^2 - 9xy + xy^2 - y^3$
Find all the second partial derivatives.

$$f_x = 4x - 9y + y^2$$
$$f_y = -9x + x(2y) - 3y^2 = -9x + 2xy - 3y^2$$

$$f_{xx} = 4$$
$$f_{yy} = 2x - 6y$$

$$f_{xy} = -9 + 2y$$
$$f_{yx} = -9 + 2y$$

Find f_{xxz} for the function $f(x, y, z) = x^3 - 4xy + 6yz - z^2$

$$f_x = 3x^2 - 4y$$
$$f_{xx} = 6x$$
$$f_{xxz} = 0$$

The other version of the derivative notation:

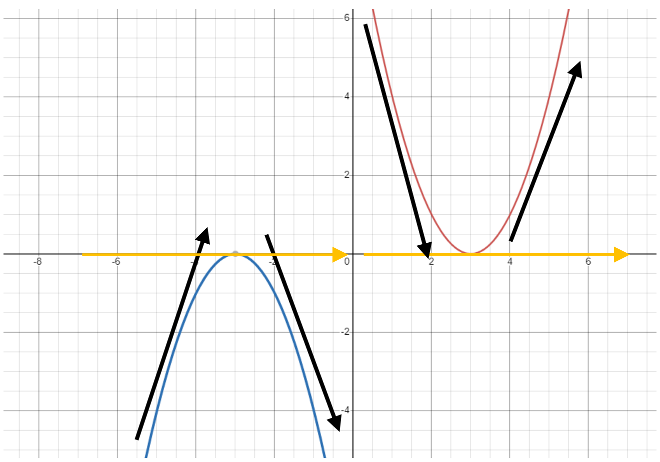
$$f_{xy} = \frac{\partial}{\partial y}\left[\frac{\partial f}{\partial x}\right] = \frac{\partial^2 f}{\partial y \partial x}$$

$$f_{xx} = \frac{\partial^2 f}{\partial x^2}$$

Comparable to $f'' = \frac{d^2 f}{dx^2}$

Optimization (one variable)

We want to be able to find the maximum and minimum of a function.

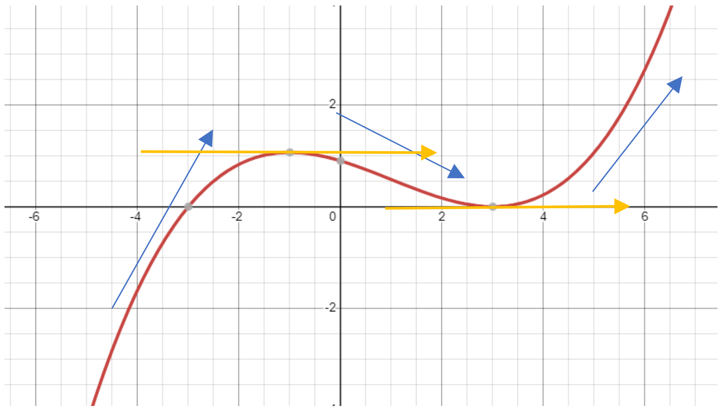


Some functions have a maximum, some have a minimum and some have both. We want to be able to find where that is.

Near a maximum, the left side of the maximum will be increasing (a positive first derivative) and to the right of the maximum, the function will be decreasing (a negative first derivative). The slope of the tangent line at the maximum is 0 (line is horizontal). Since the derivative is the slope of the tangent line, we can find the maximum by setting the derivative equal to zero.

At a minimum, the derivative will also be zero, but the slope on the left will be negative (decreasing), and the slope on the right will be increasing (positive).

These are local minima and maxima. We'll deal with the absolute case later on.



In the case of second derivatives can tell us about the way the function curves at a particular point: if it curves upward, then the second derivative is positive (call this concave up), and if the second derivative is negative, the graph curves downward (call this concave down). The point where the curvature changes direction is called an inflection point (this is where the second derivative is 0).

Critical point is the x value on a function where the first derivative is zero or undefined. These are potential maxima or minima (extrema, singular = extremum) locations. (the maximum or minimum values are obtained by plugging into the function.)

To find extrema:

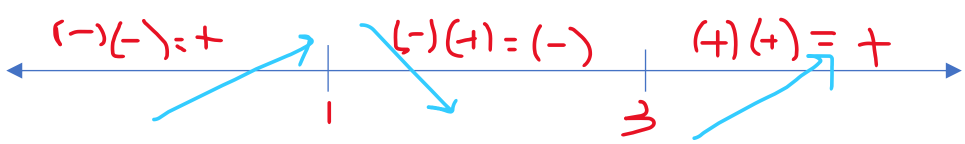
1) Take the derivative
    a. Set it equal to zero
    b. Determine if the derivative is undefined anywhere
2) Create a sign chart to look at the sign of the derivative in between the critical points, use that to determine if they are possibly maxima or minima or neither

Example. Find any extrema on the function $f(x) = x^3 - 6x^2 + 9x + 2$

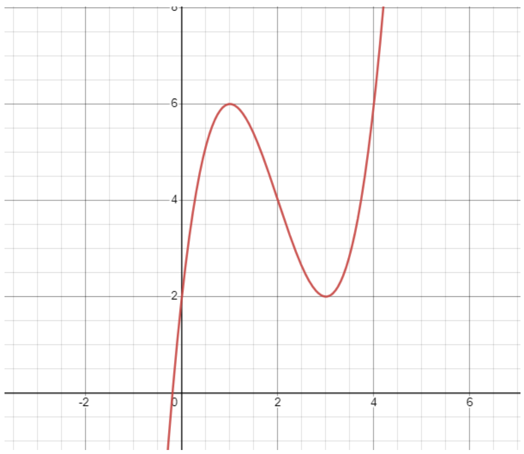
$$f'(x) = 3x^2 - 12x + 9 = 0$$
$$3(x^2 - 4x + 3) = 0$$
$$(x - 3)(x - 1) = 0$$

Critical points

$$x = 3, 1$$



The critical point x=1 should represent a maximum, and the critical point x=3, should represent a minimum.



This procedure we just went through is called the first derivative test.
There is a second derivative test.

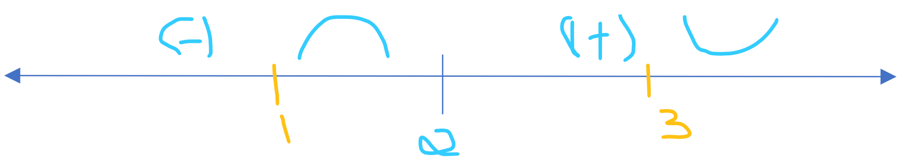
The find any extrema and characterize them:

1) Take the first derivative and set it equal to zero to find the critical points (or undefined)
2) Take the second derivative and find the inflection points (where the second derivative is zero).
3) Create a sign chart of the second derivative which tells us where the graph curves up or curves down, and locate the critical points in those regions.

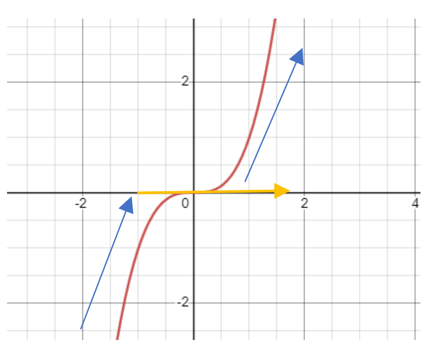
Starting with the same function:

$$f'(x) = 3x^2 - 12x + 9$$
$$x = 1,3$$

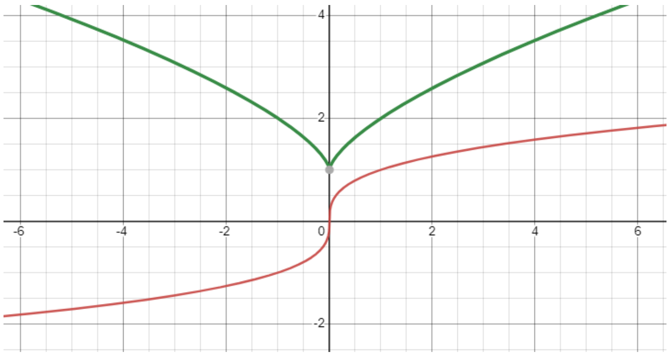
$$f''(x) = 6x - 12 = 0$$
$$x = 2$$



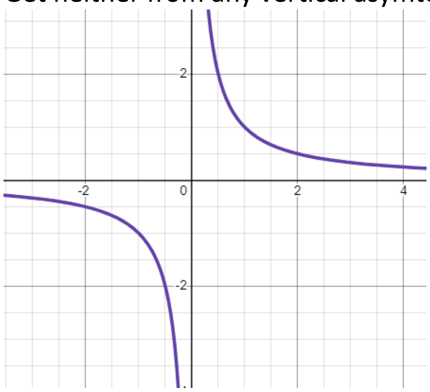
The critical point x=1, since the graph curves downward, is a maximum. And at critical point x=3, the graph curves upward, and so it represents a minimum.

Neither maximum or minimum: 

Some functions may have vertical tangents: like $y = \sqrt[3]{x}$, or there may be cusps like $y = |x|, y = \sqrt[3]{x^2}$



Get neither from any vertical asymtotes:

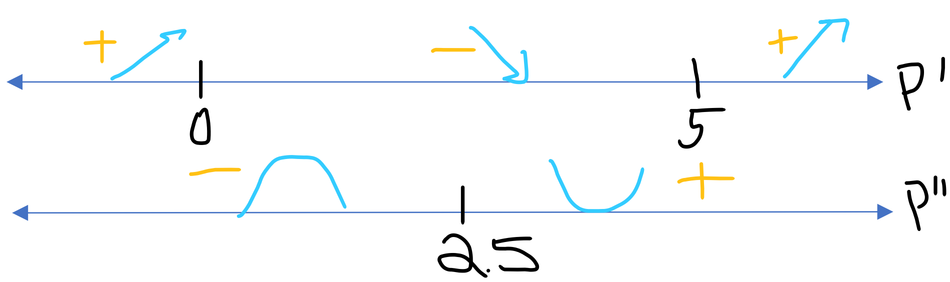


derivative is undefined at x=0, but since the function is not defined there, it can't be a maximum or a minimum.

Example.
Suppose the profit function for a company is given by $P(x) = 2x^3 - 15x^2 + 6$, where x is in thousands of units.

$$P'(x) = 6x^2 - 30x = 0$$
$$6x(x-5) = 0$$
$$x = 0,5$$

$$P''(x) = 12x - 30 = 0$$
$$x = \frac{5}{2} = 2.5$$



This says that x=0 is a maximum, and x=5 is a minimum.

Absolute extrema are guaranteed to occur on any closed interval. $[a, b]$

1) Find any critical points (set the derivative equal to zero and solve)

2) Test those critical points that are in the given interval in the function along with the end points of the interval.
3) The absolute maximum is the one with the largest y-value, and the absolute minimum is the one with the smallest y-value.

Consider the function $f(x) = 2 - x^3$ on the interval [-2,1]. Find the absolute extrema.

$$f'(x) = -3x^2 = 0$$
$$x = 0$$

Test: $f(-2), f(0), f(1)$

$$f(-2) = 2 - (-2)^3 = 2 - (-8) = 2 + 8 = 10$$
$$f(0) = 2$$
$$f(1) = 2 - (1)^3 = 2 - 1 = 1$$

Absolute max is 10 at x= -2
Absolute minimum is 1 at x=1