1/31/2024

Power Rule
Properties of Derivatives
Exponential and Logarithmic derivatives

3d plotters:
https://www.desmos.com/3d
https://www.geogebra.org/3d?lang=en
https://www.math3d.org/

Repairing a hole:

$$f(x) = \frac{(x-2)(x-3)}{x-2}$$

Domain: $(-\infty, 2) \cup (2, \infty)$

Reduce the function: $F(x) = x - 3$

F(2)= -1
Hole (2,-1)
Range: $(-\infty, -1) \cup (-1, \infty)$

Repair:

$$g(x) = \begin{cases} \frac{(x-2)(x-3)}{x-2}, x \neq 2 \\ -1, \ \ x = 2 \end{cases}$$

This makes g(x) a continuous function because there is no gap in the domain, and no gap in the range.

Numerical limit in the Excel file.

Derivative Rules
Last time we used the definition of the derivative to find the derivative functions for various functions like polynomials, rational and radical functions. We don't want to use the definition all the time if we don't have to, and it turns out there are patterns in derivatives we can exploit so that we don't have to.

The power rule.

$$\frac{d}{dx}[x^n] = n\, x^{n-1}$$

Consider x^2

$$\lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h\to 0} \frac{x^2 + 2xh + h^2 - x^2}{h} = \lim_{h\to 0} \frac{2xh + h^2}{h} = \lim_{h\to 0} \frac{h(2x+h)}{h} = 2x$$

$$\frac{d}{dx}[x^2] = 2\,x^{2-1} = 2x$$

$$(x+h)^3 = x^3 + 3xh^2 + 3x^2h + h^3$$

Consider $\frac{1}{x} = x^{-1}$

$$\frac{d}{dx}[x^{-1}] = -1\,x^{-1-1} = -1x^{-2} = -\frac{1}{x^2}$$

This does match the result we got last week using the definition.

Consider $\sqrt{x} = x^{\frac{1}{2}}$

$$\frac{d}{dx}\left[x^{\frac{1}{2}}\right] = \frac{1}{2}\,x^{\frac{1}{2}-1} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$$

This does also match the result we got last week using the definition.

The power rule will work for any real number in the exponent except 0.

$$x^0 = 1$$

Consider the function $f(x) = x^4 - 3x^{2.1} + 7x^{\frac{1}{6}} - 9x^{-5}$

One property of derivatives is that you can take the derivative term-by-term.

$$\lim_{x\to c}[f(x) + g(x)] = \lim_{x\to c} f(x) + \lim_{x\to c} g(x)$$

$$\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}[f(x)] + \frac{d}{dx}[g(x)]$$

The derivative of $f(x)$ is expressed as $f'(x)$, $\frac{df}{dx}$

$$f'(x) = 4x^3 - 3(2.1x^{1.1}) + 7\left(\frac{1}{6}x^{-\frac{5}{6}}\right) - 9(-5\,x^{-6})$$

$$f'(x) = 4x^3 - 6.3x^{1.1} + \frac{7}{6}x^{-\frac{5}{6}} + 45\,x^{-6}$$

Special Cases: linear functions and constant functions.

$$g(x) = mx + b$$

$$\lim_{h\to 0}\frac{m(x+h) + b - (mx+b)}{h} = \lim_{h\to 0}\frac{mx + mh + b - mx - b}{h} = \lim_{h\to 0}\frac{mh}{h} = m$$

The derivative of a linear function is the slope of the original line.

$$g(x) = 5x - 11$$
$$g'(x) = 5$$

$$\frac{d}{dx}[x^1] = 1x^0 = 1$$

Constant function:

$$h(x) = c$$

$$\lim_{h \to 0} \frac{c - (c)}{h} = 0$$

Recall that the derivative is the rate of change. If the function is constant, it's not changing.

$$\frac{d}{dx}[x^0] = 0x^{-1} = 0$$

Properties of derivatives.

$$\frac{d}{dx}[f(x) \pm g(x)] = \frac{d}{dx}[f(x)] \pm \frac{d}{dx}[g(x)]$$

$$\frac{d}{dx}[kf(x)] = k\frac{d}{dx}[f(x)]$$

It does not work for multiplication or division. We'll deal with rules for multiplication and division next week.

Example.

$$f(x) = \frac{x^3 - 4x^2 - 3}{x} = \frac{x^3}{x} - \frac{4x^2}{x} - \frac{3}{x} = x^2 - 4x - \frac{3}{x} = x^2 - 4x - 3x^{-1}$$

Find the derivative.

$$f'(x) = 2x - 4 + 3x^{-2} = 2x - 4 + \frac{3}{x^2}$$

Applications, word problems.

Consider a particle moving along a path defined by $f(x) = 2x^2 - x$. Find the **instantaneous rate of change** of the particle at $x = 3$

Instantaneous rate of change means take the derivative and plug in the point.

$$f'(x) = 4x - 1$$
$$f'(3) = 11$$

Consider a particle moving along a path defined by $f(t) = 2t^2 - t$. Find the **instantaneous rate of change** of the particle at $t = 3$.

A fence is defined by a function $f(x) = x^{\frac{2}{3}}$. Find the **slope of the tangent line** when $x = 8$.

Slope of the tangent line means take the derivative and plug in the value.

$$f'(x) = \frac{2}{3}x^{-\frac{1}{3}} = \frac{2}{3\sqrt[3]{x}}$$

$$f'(8) = \frac{2}{3\sqrt[3]{8}} = \frac{1}{3}$$

A fence is defined by a function $f(x) = x^{\frac{2}{3}}$. Find the **equation of the tangent line** when $x = 8$

First find the slope of the tangent line, and find the point on the original function at the given value.

$$f(x) = \sqrt[3]{x^2} = x^{\frac{2}{3}}$$

$$f(8) = \left(\sqrt[3]{8}\right)^2 = 2^2 = 4 = \sqrt[3]{8^2} = \sqrt[3]{64} = 4$$

The line passes through the point (on the curve) (8,4)

$$y - 4 = \frac{1}{3}(x - 8)$$

$$y - 4 = \frac{1}{3}x - \frac{8}{3}$$

$$y = \frac{1}{3}x + \frac{4}{3}$$

Marginal = derivative
Marginal cost = derivative of the cost
Velocity = derivative of the position
Acceleration = derivative of the velocity

Logarithmic and Exponential Function derivatives.

$$\log x = \log_{10} x$$

$$\ln x = \log_e x$$

$$\frac{d}{dx}[\ln x] = \frac{1}{x}$$

$$\frac{d}{dx}[\log_a x] = \frac{d}{dx}\left[\frac{\ln x}{\ln a}\right] = \frac{1}{\ln a} \times \frac{d}{dx}[\ln x] = \frac{1}{\ln a} \times \frac{1}{x} = \frac{1}{x \ln a} = \frac{1}{(\ln a)x}$$

$$\frac{d}{dx}[e^x] = e^x$$

$$\frac{d}{dx}[a^x] = (\ln a)a^x$$

$$a^x = \left(e^{\ln a}\right)^x = e^{x \ln a}$$

$$\frac{d}{dx}\left[e^{x \ln a}\right] = \ln a \left[e^{x \ln a}\right] = \ln(a)\, a^x$$

Example.
Find the derivative of $f(x) = \ln x + e^x$

$$f'(x) = \frac{1}{x} + e^x$$

Find the derivative of $g(x) = \ln(4x^2) + 6e^{x+2}$

$$\ln(4x^2) = \ln 4 + \ln x^2 = \ln 4 + 2 \ln x$$

$$6e^{x+2} = 6e^x e^2 = (6e^2)e^x$$

$$g'(x) = 2\left(\frac{1}{x}\right) + (6e^2)e^x$$

Find the derivative of $h(x) = \log x + 2^x$

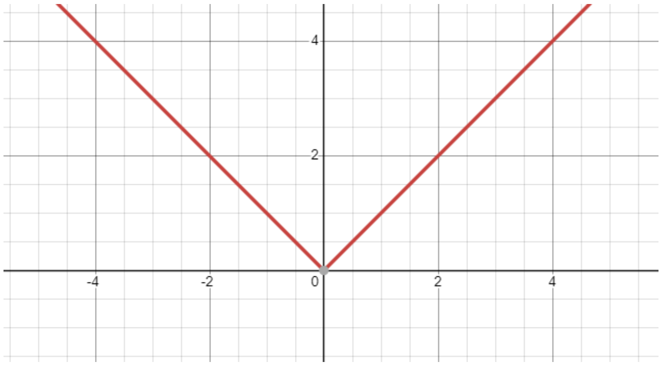
$$h'(x) = \frac{1}{(\ln 10)x} + (\ln 2)2^x$$

Continuity and differentiability

When a function has a derivative, it is said to be differentiable.
If a function is differentiable (on some interval), then the function is continuous (on its domain)

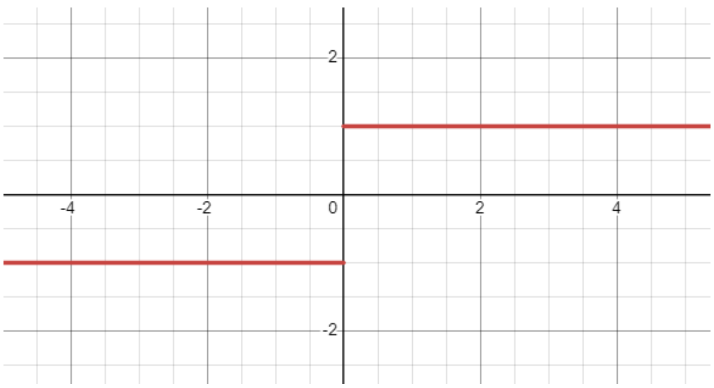
If a function is continuous, cannot necessarily conclude that it is differentiable—continuous derivative).

$$f(x) = |x| = abs(x)$$



$$f(x) = \begin{cases} x, x \geq 0 \\ -x, x < 0 \end{cases}$$

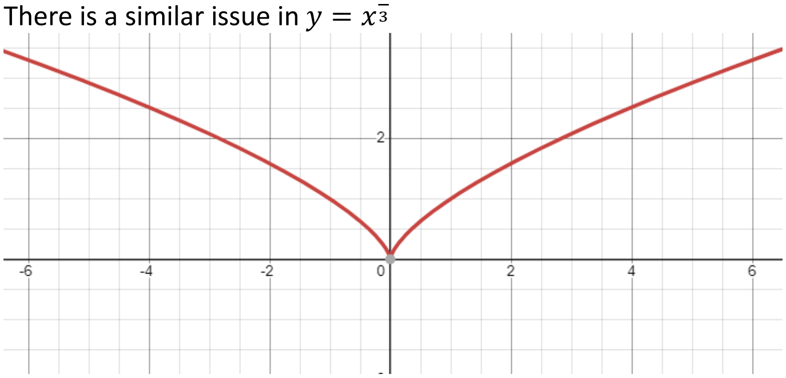
The derivative of the right side is 1, and the derivative of the left side is -1



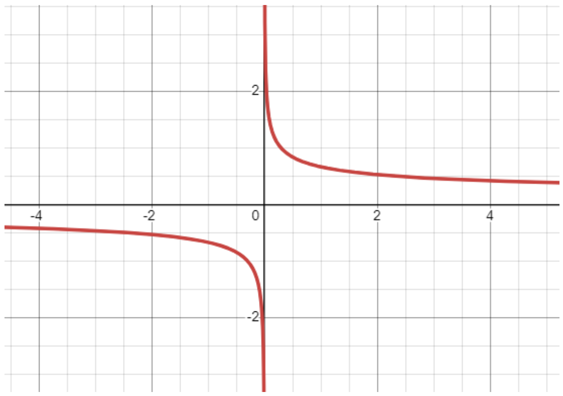
This is not a continuous derivative because the derivative (limit) is not defined at x=0. This function is not differentiable at x=0.
The original function is continuous at x=0.

The issue here is the "cusp" – that pointy bit in the graph.

There is a similar issue in $y = x^{\frac{2}{3}}$



$$y' = \frac{2}{3}x^{-\frac{1}{3}} = \frac{2}{3\sqrt[3]{x}}$$



Piecewise functions often have these properties.