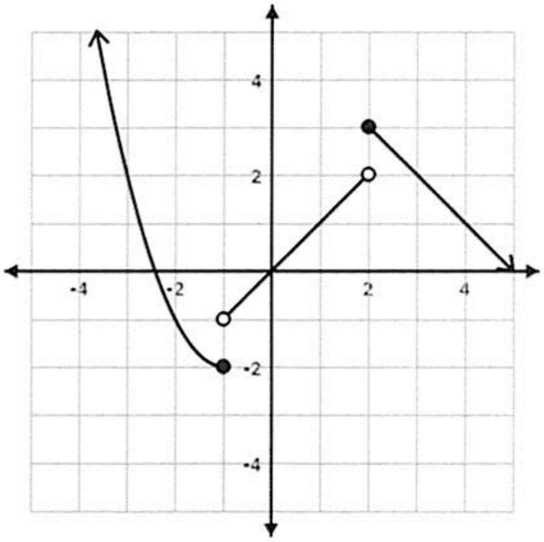
MTH 261, Final Exam, Spring 2024 Name KEY

Instructions: Work the problems below as directed. Show all work. Clearly mark your final answers. Use exact values unless the problem specifically directs you to round. Simplify as much as possible. Partial credit is possible, but solutions without work will not receive full credit.

Part 1: These questions you will submit answers to in Canvas. Show all work and submit the work with Part 2 of the exam. But you must submit the answers in Canvas to receive credit. Each question/answer will be listed separately. The Canvas question will refer to the number/part to indicate where you should submit which answer. The questions will appear in order (in case there is an inadvertent typo). Correct answers will receive full credit with or without work in this section, but if you don't submit work and clearly label your answers, you won't be able to challenge any scoring decisions for making an error of any kind.

1. Use the graph below to answer the following questions. (4 points each)
   a. $\lim_{x\to 2^+} F(x)$ 3
   b. $\lim_{x\to 2^-} F(x)$ 2
   c. $\lim_{x\to 2} F(x)$ DNE
   d. $F(2)$ 3



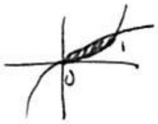
   e. Describe the discontinuity at $x = 2$. Is it a jump discontinuity or a removable discontinuity? If it is removable, what value would $F(2)$ need to be to make the graph continuous at that point?

jump

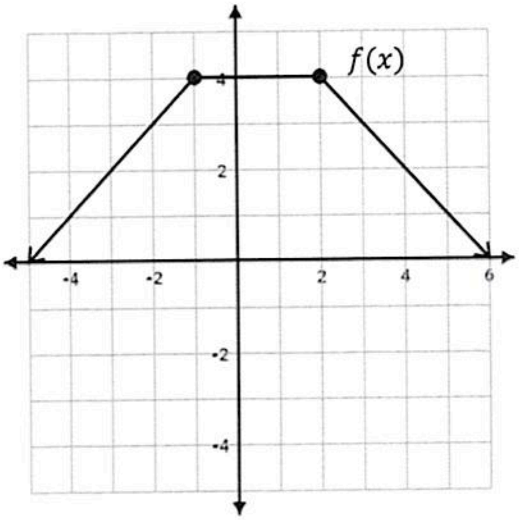
2. Find the area bounded by $f(x) = x^3$ and $g(x) = \sqrt{x}$. Sketch the graph. (8 points)

$$\int_0^1 x^{1/2} - x^3\, dx = \frac{2}{3}x^{3/2} - \frac{1}{4}x^4 \Big|_0^1$$

$$\frac{2}{3} - \frac{1}{4} = \frac{5}{12}$$



3. Use graph to the right to evaluate the integral $\int_{-5}^{6} f(x)dx$ geometrically. (8 points)



$A = \frac{1}{2}(4)(11+3) =$

$2(14) = 28$

4. Integrate. (8 points each)

a. $\int x^3 e^x dx$

$x^3e^x - 3x^2e^x + 6xe^x - 6e^x + C$

±	u	dv
+	x^3	e^x
−	$3x^2$	e^x
+	$6x$	e^x
−	6	e^x
	0	e^x

b. $\int \frac{(\ln x)^2}{x} dx$

$u = \ln x$

$du = \frac{1}{x} dx$

$\int u^2 \, du = \frac{1}{3}u^3 + C = \frac{1}{3}(\ln x)^3 + C$

5. Riverside Appliances is marketing a new mini-refrigerator. It determines that in order to sell x refrigerators, the price per suit must be $p = 200 - x$. It also determines that the total cost of producing x refrigerators is given by $C(x) = \frac{1}{3}x^3 - 6x^2 + 89x + 100$.

a. Find the total revenue $R(x)$. (3 points)

$$R(x) = 200x - x^2$$

b. Find the total profit $P(x)$. (3 points)

$$P = R - C$$
$$= 200x - x^2 - \frac{1}{3}x^3 + 6x^2 - 89x - 100 = -\frac{1}{3}x^3 + 5x^2 + 111x - 100$$

c. How many refrigerators must the company produce and sell in order to maximize profit? (5 points)

$$P'(x) = -3x^2 + 10x + 111 = 0$$
$$\frac{-10 \pm \sqrt{100 - 4(-3)(111)}}{-6} = \frac{-10 \pm \sqrt{5428}}{-6} \approx 13.94 \rightarrow 14$$

d. What is the maximum profit? (4 points)

$$P(14) = \$1519.3$$

e. What is the price per refrigerator that must be charged in order to maximize profit? (4 points)

$$200 - 14 = \$186$$

6. Carbon-14 has a decay rate that is modeled by the equation $\frac{dN}{dt} = -0.00012097N$, where t is in years. How old is a fire pit if 18% of its original Carbon-14 remains? (10 points)

$$\frac{\ln(0.18)}{-0.00012097} = 14{,}175.4 \text{ years}$$

7. The elasticity of demand is given by $E(x) = -\frac{xD'(x)}{D(x)}$. Find the elasticity for $D(x) = \frac{400}{x}$, at $x = 75$. (8 points)

$$D' = \frac{-400}{x^2}$$

$$E(x) = \frac{-\cancel{x}\left(\frac{-400}{\cancel{x^2}}\right)}{\frac{\cancel{400}}{\cancel{x}}} = 1$$

8. Approximate the area under the curve $f(x) = \frac{1}{x^2}$ on the interval [1,4] by computing the area under 6 rectangles (using the left-hand rule). (15 points)

$$\Delta x = \frac{4-1}{6} = \frac{3}{6} = \frac{1}{2}$$

$x_0 = 1,\ x_1 = 1.5,\ x_2 = 2,\ x_3 = 2.5,\ x_4 = 3,$
$x_5 = 3.5,\ x_6 = 4$

$A_i = f(x_i)\Delta x \quad \Sigma f(x_i)\Delta x = A$

$$\approx \frac{1}{2}\left[1 + \frac{1}{1.5^2} + \frac{1}{2^2} + \frac{1}{2.5^2} + \frac{1}{3^2} + \frac{1}{3.5} + \frac{1}{4^2}\right] = 1.054844..$$

9. Find the accumulated present value of a continuous income stream if $B(t) = \int_0^T R(t)e^{-kt}dt$, where $R(t) = 3000t + 7, T = 40, k = 7\%$. (10 points)

$$\int_0^{40} (3000t+7)e^{-0.07t} \qquad u = 3000t+7 \quad dv = e^{-0.07t}dt$$

$$du = 3000\,dt \quad v = -\frac{1}{0.07}e^{-0.07t}$$

$$\frac{(3000t+7)}{-0.07}e^{-0.07t} - \frac{3000}{(-0.07)^2}e^{-0.07t}\Big|_0^{40} = 470,862.34$$

10. Find the value of k so that $f(x) = \frac{k}{x}$ is a probability density function on the interval $[1,3]$. (10 points)

$$\int_1^3 \frac{k}{x}dx = k\ln x\Big|_1^3 = k(\ln 3 - \overset{0}{\cancel{\ln 1}}) = 1 \rightarrow$$

$$K = \frac{1}{\ln 3}$$

11. Given $f(x) = \frac{3}{125}x^2$ is a probability density function on $[0,5]$, find the following:

a. $P(1 \le x \le 4)$ (8 points)

$$P(1 \le x \le 4) = \int_1^4 \frac{3}{125}x^2dx = \frac{1}{125}x^3\Big|_1^4 = \frac{1}{125}[64-1] =$$

$$\frac{63}{125}$$

b. μ (mean) $= \int_a^b xf(x)dx$ (8 points)

$$\frac{3}{125}\int_0^5 x^3\,dx = \frac{3}{125}\cdot\frac{1}{4}x^4\Big|_0^5 = \frac{3}{500}\left[625\right] = 3.75$$

Part 2: In this section you will record your answers on paper along with your work. After scanning, submit them to a Canvas dropbox as directed. These questions will be graded by hand.

12. Use the definition of the derivative $f'(x) = \lim\limits_{h\to 0}\frac{f(x+h)-f(x)}{h}$ to find the derivative of the functions $f(x) = -x^2 + 3x - 8$. (10 points)

$$\lim_{h\to 0}\frac{-\cancel{x^2}-2xh-h^2+\cancel{3x}+3h\cancel{-8}+\cancel{x^2}-\cancel{3x}\cancel{+8}}{h} = \lim_{h\to 0}\frac{-2xh-h^2+3h}{h}$$

$$= \lim_{h\to 0}\frac{\cancel{h}(-2x-h+3)}{\cancel{h}} = -2x+3$$

13. Find f_{xy} and f_{yy} for the function $f(x,y) = x^3y^2 - x^2y^2$. (10 points)

$f_x = 3x^2y^2 - 2xy^2$

$f_{xy} = 6x^2y - 4xy$

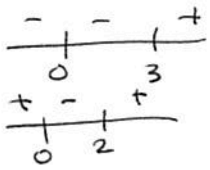
$f_y = 2x^3y - 2x^2y$

$f_{yy} = 2x^3 - 2x^2$

14. Use the first and second derivative tests to determine whether each of the critical points of the graph $g(x) = x^4 - 4x^3 + 10$ is a maximum or a minimum (or neither). Your supporting work should show sign charts for both derivatives and identify any inflection points. (12 points)

$g'(x) = 4x^3 - 12x^2 = 0 \quad 4x^2(x-3) = 0$

$g''(x) = 12x^2 - 24x = 0 \quad 12x(x-2) = 0$



0 is neither (inflection point)

3 is a minimum

15. Sketch the graph of the function $f(x) = \frac{1}{x^2+1}$ using calculus and algebraic techniques. Note any intercepts, critical points, and asymptotes. (12 points)

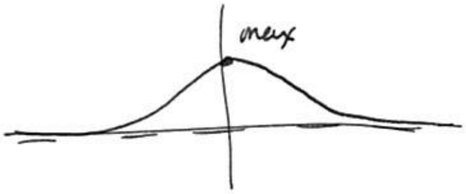
horizontal asymptote : $y=0$

no VA or holes

$f(x) = (x^2+1)^{-1}$

$f'(x) = -(x^2+1)^{-2} \cdot 2x$

$= \frac{-2x}{(x^2+1)^2}$ Critical point at $x=0$



16. Find the equation of the tangent line to the implicitly defined function $x^4 - x^2y^3 = 12$ at the point $(-2,1)$. (10 points)

$4x^3 - 2xy^3 - 3x^2y^2\,y' = 0$

$\frac{4x^3 - 2xy^3}{3x^2y^2} = \frac{dy}{dx} \rightarrow \frac{4(-8) - 2(-2)(1)}{3(4)(1)} = \frac{-32+4}{12} = \frac{-28}{12} = \frac{-7}{3}$

$y - 1 = \frac{-7}{3}(x+2)$

17. Integrate $\int \frac{4}{\sqrt[3]{x^2}} + \frac{3}{5}e^{-2x} - \frac{8}{x}dx$. (8 points)

$4x^{-2/3}$

$$3\cdot 4x^{1/3} + \frac{3}{-10}e^{-2x} - 8\ln x + C$$

$$12\sqrt[3]{x} - \frac{3}{10}e^{-2x} - 8\ln x + C$$

18. Find the derivative of the functions below. (8 points each)

a. $g(x) = e^{x^2} - \frac{4}{3}e^{x-1} + (1.5)^x$

$$g'(x) = 2xe^{x^2} - \frac{4}{3}e^{x-1} + (\ln 1.5)1.5^x$$

b. $h(x) = x[\ln(e^x - 1)]^3$

$$h'(x) = [\ln(e^x-1)]^3 + x\left[3[\ln(e^x-1)]^2\cdot\frac{1}{e^x-1}\cdot e^x\right]$$

c. $F(x) = \frac{8x+\ln x}{x^2-4}$

$$F(x) = \frac{(8+\frac{1}{x})(x^2-4) - 2x(8x+\ln x)}{(x^2-4)^2}$$

19. Find the first 5 derivatives of $g(x) = \frac{1}{3}x^8 - 5x^6 - 2x^4 - 9x^2 + 7$. (10 points)

$g'(x) = \frac{8}{3}x^7 - 30x^5 - 8x^3 - 18x$

$g''(x) = \frac{56}{3}x^6 - 150x^4 - 24x^2 - 18$

$g'''(x) = 112x^5 - 600x^3 - 48x$

$g^{iv}(x) = 560x^4 - 1800x^2 - 48$

$g^{v}(x) = 2240x^3 - 3600x$

20. Find the first partial derivatives of the functions $g(x, y, z) = e^{zy^2} - x \ln z + x^e y^{0.1}$. (9 points)

$g_x = \ln z + ex^{e-1}y^{0.1}$

$g_y = e^{zy^2} \cdot 2yz + x^e \, 0.1y^{-0.9}$

$g_z = e^{zy^2} \cdot y^2 - \frac{x}{z}$

21. Find the critical point(s) and determine whether each is a maximum, minimum, saddle point, or cannot be determined, for the function $f(x, y) = 2xy - x^3 - y^2$. (15 points)

$f_x = 2y - 3x^2 = 0$ $\quad 2x - 3x^2 = 0 \quad x(2 - 3x) = 0$

$f_y = 2x - 2y = 0 \quad x = y \quad x = 0 \quad x = \frac{2}{3}$

$(0,0) \quad (2/3, 2/3)$

↙

$f_{xx} = -6x$

$f_{yy} = -2$

$f_{xy} = 2$

$D = 0(-2) - 2^2 = -4$ saddle

$D = -6(2/3)(-2) - 2^2 =$

$8 - 4 = 4 > 0$ maximum

$f_{xx} < 0 \cap$

22. Find the consumer's and producer's surplus for $D(x) = \frac{900}{\sqrt{x+1}}, S(x) = \sqrt{x+1}$. (15 points)

$$\frac{900}{\sqrt{x+1}} = \sqrt{x+1} \qquad D(899) = \frac{900}{\sqrt{900}} = \frac{900}{30} = 30$$

$$900 = x + 1$$

$$x = 899$$

$$\int_0^{899} \frac{900}{\sqrt{x+1}} - 30\,dx = 2 \cdot 900\sqrt{x+1} - 30x \Big|_0^{899}$$

$$1800 \cdot 30 - 30 \cdot 899 - 1800(1) = 25{,}230$$

$$\int_0^{899} 30 - \sqrt{x+1}\,dx = 30x - \frac{2}{3}(x+1)^{3/2} \Big|_0^{899}$$

$$30(899) - \frac{2}{3}(30)^3 + \frac{2}{3}(1) =$$

$$\frac{26912}{3} \approx 8970.67$$