4/7/2022

Integrating rational functions (part of 5.6)
Integrating inverse trig functions (5.7)
Integrating hyperbolic trig functions (part of 6.9)

A rational function we can use log rules with

$$\int \frac{2x^3+3x}{x^4+3x^2}dx$$

The thing to check when you have a fraction: is the numerator the derivative (or a constant multiple of the derivative) of the denominator?

$$u = x^4 + 3x^2$$
$$du = (4x^3+6x)dx = 2(2x^3+3x)dx$$

$$\frac{1}{2}du = (2x^3+3x)dx$$

$$\int \frac{2x^3+3x}{x^4+3x^2}dx = \int \frac{\frac{1}{2}du}{u} = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln|x^4+3x^2| + C$$

If the degree of the numerator is NOT less than the denominator then none of the rules we are talking about are going to apply. You'll have to do long division first.

Inverse Trig Functions

$$\int \frac{du}{\sqrt{a^2-u^2}} = \sin^{-1}\left(\frac{u}{a}\right) + C$$

$$\int \frac{du}{u^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C$$

$$\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}\sec^{-1}\left(\frac{u}{a}\right) + C$$

Example.

$$\int \frac{dx}{\sqrt{4-9x^2}}$$

$$a = 2, u = 3x$$
$$du = 3dx$$
$$\frac{1}{3}du = dx$$

$$\int \frac{dx}{\sqrt{4-9x^2}} = \int \frac{\frac{1}{3}du}{\sqrt{4-u^2}} = \frac{1}{3}\int \frac{du}{\sqrt{2^2-u^2}} = \frac{1}{3}\arcsin\left(\frac{3x}{2}\right) + C$$

Example.

$$\int \frac{x}{16+x^4}dx$$

If I let $u = 16 + x^4$, then $du = 4x^3dx$ and I don't have an x^3.
Is the denominator the sum of squares? If so, it might be an inverse tangent. Let $u = x^2$ since $(x^2)^2 = x^4$ and let $a = 4$. What is du? $du = 2xdx$

$$\frac{1}{2}du = xdx$$

$$\int \frac{x}{16+x^4}dx = \int \frac{\frac{1}{2}du}{4^2+u^2} = \frac{1}{2}\int \frac{du}{4^2+u^2} = \frac{1}{2}\left[\frac{1}{4}\arctan\frac{u}{a}\right] + C = \frac{1}{8}\arctan\frac{x^2}{4} + C$$

Example.

$$\int \frac{dx}{x\sqrt{4x^2-9}}$$

$$u = 2x, du = 2dx$$

$$\int \frac{dx}{x\sqrt{4x^2-9}} = \int \frac{2dx}{2x\sqrt{4x^2-9}} = \int \frac{du}{u\sqrt{u^2-3^3}} = \frac{1}{3}\text{arcsec}\left(\frac{u}{3}\right) + C = \frac{1}{3}\text{arcsec}\left(\frac{2x}{3}\right) + C$$

Example.

$$\int \frac{\cos x}{9+\sin^2 x}dx$$

$$u = \sin x\,, du = \cos x\, dx, a = 3$$

$$\int \frac{\cos x}{9+\sin^2 x}dx = \int \frac{du}{3^2+u^2} = \frac{1}{3}\arctan\left(\frac{\sin x}{3}\right) + C$$

Example.

$$\int \frac{e^x}{(1+e^{2x})}dx$$
$$u = e^x, du = e^x dx$$

Example.

$$\int \frac{\cos^{-1} x}{\sqrt{1-x^2}}dx$$

$$recall: \frac{d}{dx}[\cos^{-1} x] = -\frac{1}{\sqrt{1-x^2}}$$

$$u = \cos^{-1} x \,, du = -\frac{1}{\sqrt{1-x^2}} dx$$

Example.

$$\int \frac{e^t \arctan(e^t)}{1+e^{2t}} dt$$

$$u = \arctan e^t \,, du = \frac{1}{1+e^{2t}} e^t dt$$

$$\int \frac{e^t \arctan(e^t)}{1+e^{2t}} dt = \int u \, du = \frac{1}{2}u^2 + C = \frac{1}{2}(\arctan e^t)^2 + C$$

If you see an inverse trig function inside an integral, let that be u… that usually works.
If you see a function inside an integral for which we don't have an antiderivative rule, try letting u be that function.

Hyperbolic trig functions

$$\int \sinh x \, dx = \cosh x + C$$
$$\int \cosh x \, dx = \sinh x + C$$

$$\int \operatorname{sech}^2 x \, dx = \tanh x + C$$

$$\int \operatorname{csch}^2 x \, dx = -\coth x + C$$

$$\int \operatorname{sech} x \tanh x \, dx = -\operatorname{sech} x + C$$

$$\int \operatorname{csch} x \coth x \, dx = -\operatorname{csch} x + C$$

Use substitution to do $\int \tanh x \, dx$.

Example.

$$\int \sinh^3 x \cosh x \, dx$$
$$u = \sinh x \,, du = \cosh x \, dx$$

$$\int \sinh^3 x \cosh x \, dx = \int u^3 du = \frac{1}{4}u^4 + C = \frac{1}{4}\sinh^4 x + C$$

What do we do with definite integrals using substitution… what happens to the limits of integration?

Example.

$$\int_0^{\frac{\pi}{4}} \frac{sin\theta}{\cos^4\theta} d\theta$$

$$u = \cos\theta\,, du = -\sin\theta\, d\theta$$
$$-du = sin\theta d\theta$$

If we don't want to change the limits of integration: integrate to find the function and then go back to theta.

$$\int \frac{\sin\theta\, d\theta}{\cos^4\theta} = \int -\frac{du}{u^4} = -\int u^{-4}du = \frac{u^{-3}}{3} + C = \frac{1}{3} \times \frac{1}{\cos^3\theta} + C = \frac{1}{3}\sec^3\theta + C$$

$$\int_0^{\frac{\pi}{4}} \frac{sin\theta}{\cos^4\theta} d\theta = \frac{1}{3}\sec^3\theta\Big|_0^{\pi/4} = \frac{1}{3}\left[\,2\sqrt{2} - 1\right]$$

Or, replace limits in terms of u and then don't go back to theta.

$$\int_0^{\frac{\pi}{4}} \frac{sin\theta}{\cos^4\theta} d\theta = \int_1^{\frac{1}{\sqrt{2}}} -u^{-4}du = \frac{u^{-3}}{3}\Bigg|_1^{\frac{1}{\sqrt{2}}} = \frac{1}{3}\left[\left(\frac{1}{\sqrt{2}}\right)^{-3} - 1^{-3}\right] = \frac{1}{3}\left[\,\left(\sqrt{2}\right)^3 - 1\right]$$

When $\theta = \frac{\pi}{4}, u = \cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$
When $\theta = 0, u = \cos(0) = 1$