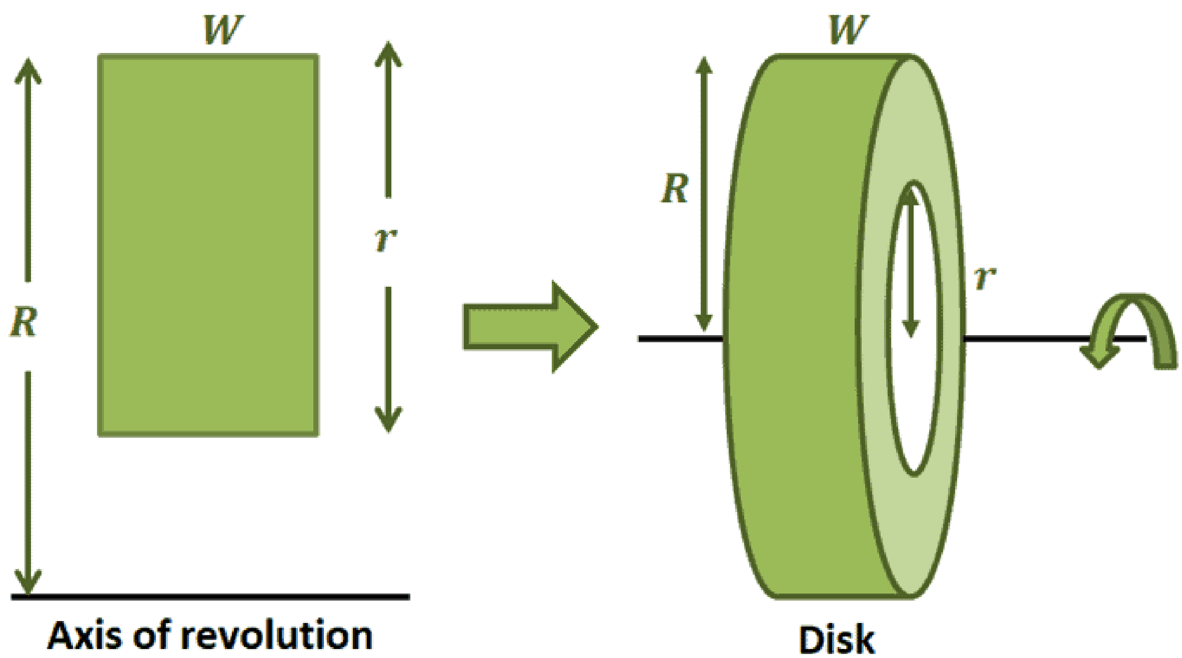
4/21/2022

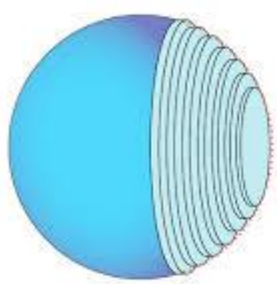
Volumes of Revolution (6.2/6.3)

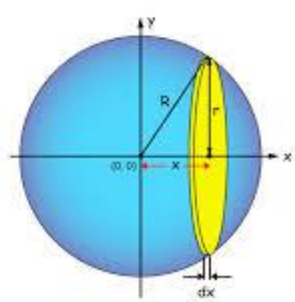
These cover the idea of rotating a 2d function (y=f(x) or x=f(y)) around one of the axes. Conceptually, we are thinking about these shapes similarly to the way we think about them in terms of Riemann sums.

Washer Method (Disk Method)
Shell Method

Washer Method (6.2)







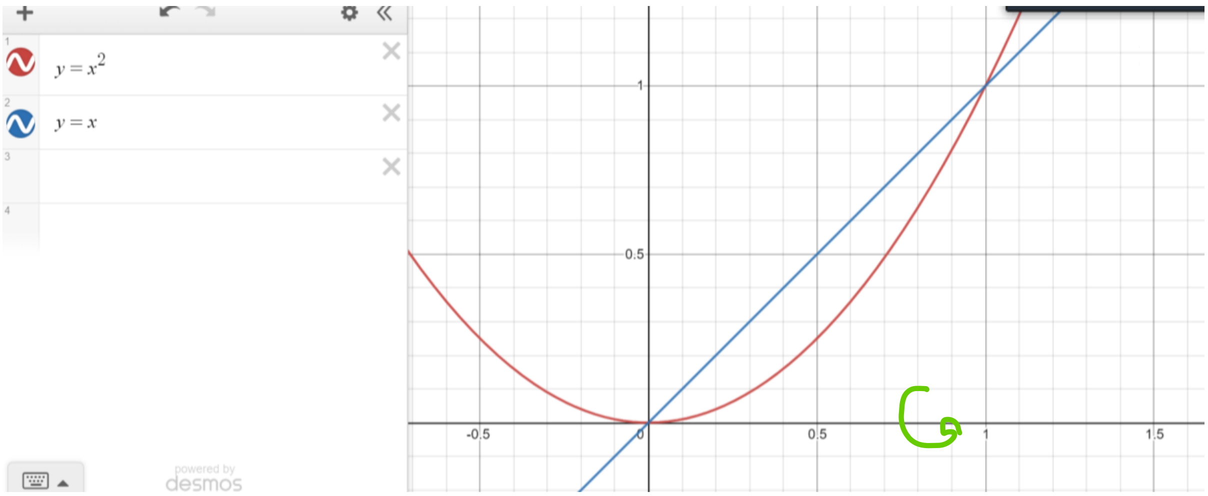
Each disk/washer $R_{outer}(x), R_{inner}(x)$
A solid cylinder has the formula: $V = \pi r^2 h$
Volume of one disk: $\pi\left(f(x)\right)^2 \Delta x$
Disk Method: add up the slices and let Δx get really small… $\int_a^b \pi f^2(x)dx$

$$V = \pi \int_a^b [f(x)]^2 dx$$

Washer Method: there is an inner function, so the Volume of the washer has to subtract out the volume of the hole.

A washer has the formula: $V = \pi r_{outer}^2 h - \pi r_{inner}^2 h$
Washer method:

$$V = \pi \int_a^b \left[R_{outer}^2 - R_{inner}^2\right] dx$$

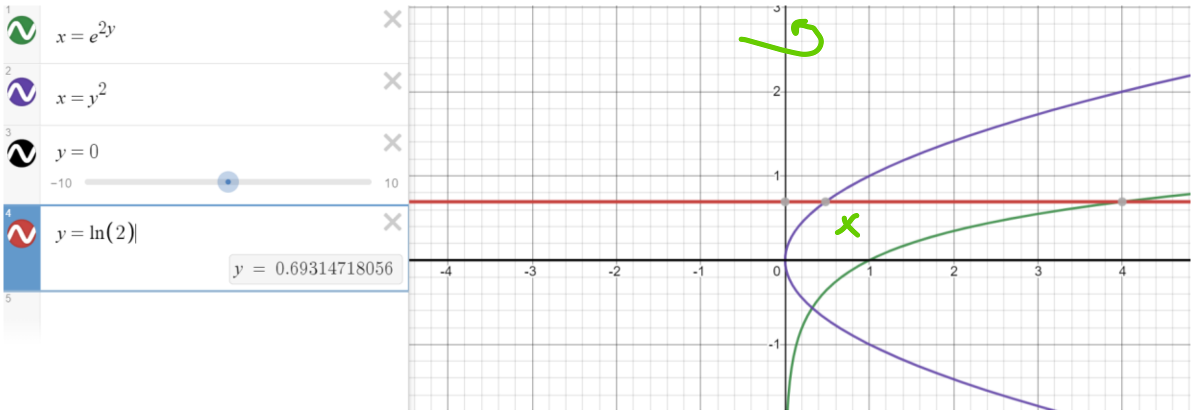


(Use the washer method)
Find the volume of the solid of revolution obtained from rotating the region between $y = x^2$ and $y = x$ around the x-axis.

$$V = \pi \int_0^1 [x^2 - (x^2)^2] dx$$

$$= \pi \int_0^1 x^2 - x^4 dx = \pi \left[\frac{1}{3}x^3 - \frac{1}{5}x^5\right]_0^1 = \pi \left(\frac{1}{3} - \frac{1}{5}\right) = \frac{2\pi}{15}$$

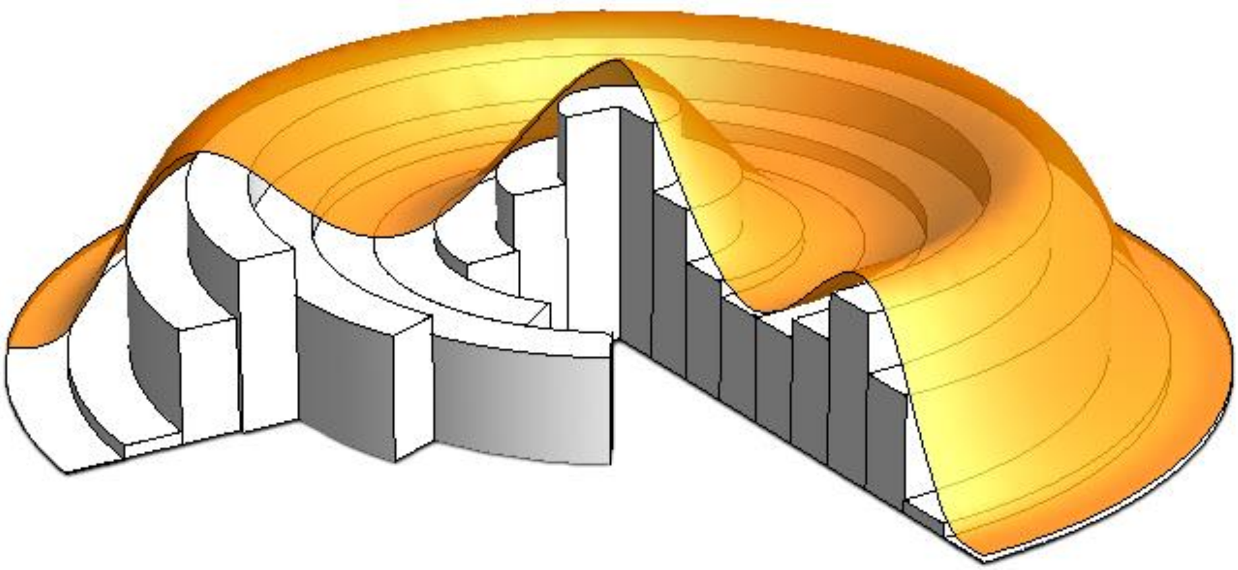
Another example.
Find the volume of the solid of revolution obtained from rotating the region bounded by $x = e^{2y}, x = y^2, y = 0, y = \ln(2)$ around the y-axis.
When you are using the washer or disk method, the variable that you integrate with is the same as the axis you rotate around.

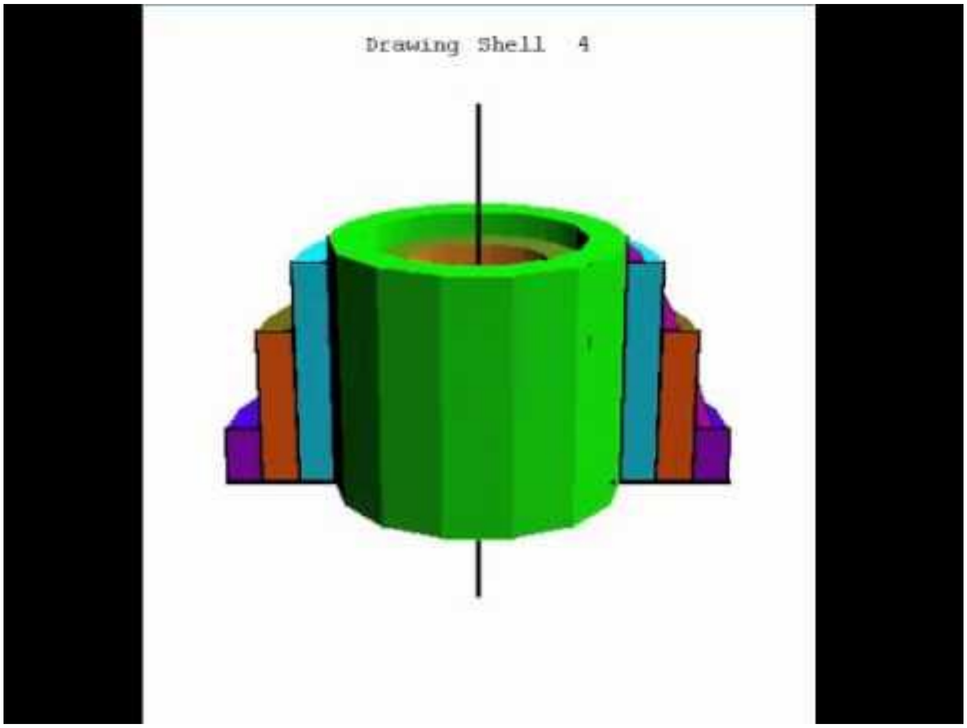


$$V=\pi\int_c^d\left[R_{outer}^2-R_{inner}^2\right]dy=\pi\int_0^{\ln 2}(e^{2y})^2-(y^2)^2dy=\pi\int_0^{\ln 2}e^{4y}-y^4dy$$

$$=\pi\left[\frac{1}{4}e^{4y}-\frac{1}{5}y^5\right]_0^{\ln 2}=\pi\left[\frac{1}{4}(16)-\frac{1}{5}(\ln 2)^5-\frac{1}{4}\right]=\pi\left[\frac{15}{14}-\frac{(\ln 2)^5}{5}\right]$$

Shell Method
The variable you integrate with is the opposite of the axis you are rotating around.
If you rotate around the y-axis, integrate with functions of x.



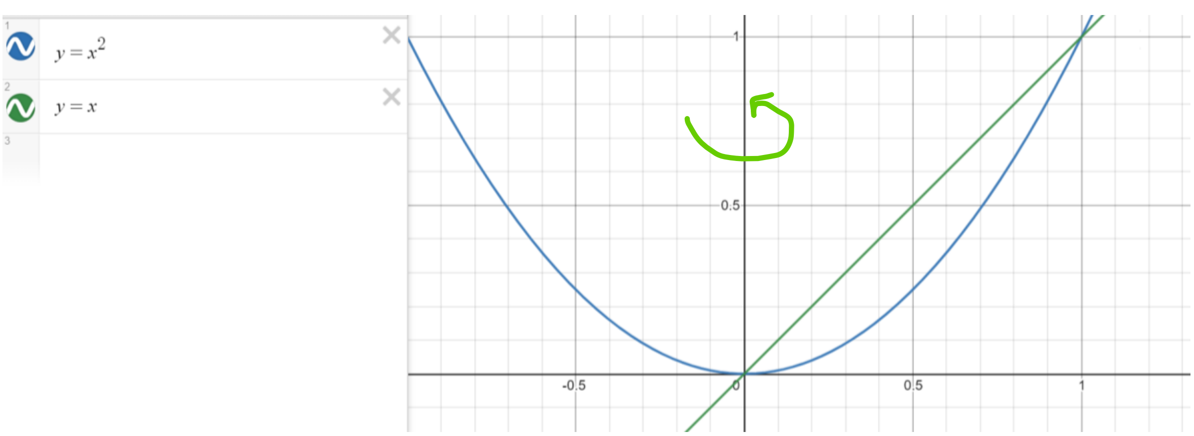


What is the volume of each particular cylinder?
Imagine that we unwrapped this cylinder. Thickness is the thickness of the rectangle Δx.
So we can multiply the area (surface area) by the thickness, you get volume.
The unwrapped cylinder is a rectangle. The area is the height of the cylinder (the function). The circumference of the circle on the base is the width of the rectangle. $C = 2\pi r$.
What's r? It's the distance to the axis of rotation. In this configuration, r=x.

$$V = 2\pi x[f(x)]\Delta x$$

Let our Δx get really small, and add up all the shells:

$$V = 2\pi \int_a^b x f(x) dx$$



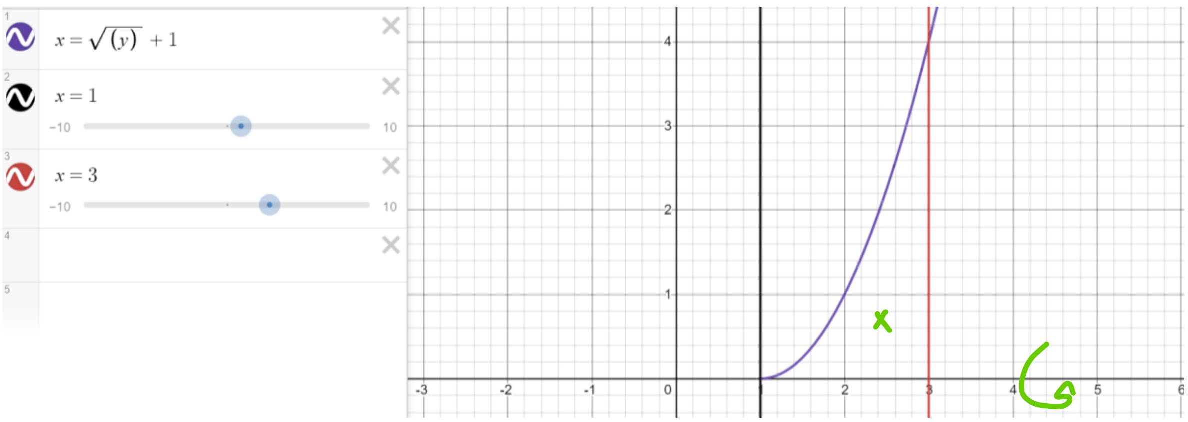
Find the volume of revolution from rotating the region bounded by $y = x^2$ and $y = x$ around the y-axis.

$$V = 2\pi \int_0^1 x[f(x) - g(x)]dx = 2\pi \int_0^1 x[x - x^2]dx = 2\pi \int_0^1 x^2 - x^3 dx =$$

$$2\pi \left[\frac{1}{3}x^3 - \frac{1}{4}x^4\right]_0^1 = 2\pi \left[\frac{1}{3} - \frac{1}{4}\right] = \frac{\pi}{6}$$

Around the x-axis using shells

Find the volume of revolution from rotating the region bounded by $x = \sqrt{y} + 1, x = 1, x = 3$ around the x-axis.



Using shells

Top function (rightmost one) is x=3, bottom function (leftmost one) is $x = \sqrt{y} + 1$,

Radius is in y.

Since we are integrating in y, we need limits in y.

X=1 corresponds to y=0, x=3 corresponds to y=4

$$V = 2\pi \int_0^4 y[3 - (\sqrt{y} + 1)]dy = 2\pi \int_0^4 y(2 - \sqrt{y})dy = 2\pi \int_0^4 2y - y^{\frac{3}{2}}dy$$

$$= 2\pi \left[y^2 - \frac{2}{5}y^{\frac{5}{2}}\right]_0^4 = 2\pi \left[16 - \frac{2}{5}(32)\right] = 2\pi \left[16 - \frac{64}{5}\right] = \frac{32\pi}{5}$$

What if we did this in washer or disk method?

$$x = \sqrt{y} + 1 \to x - 1 = \sqrt{y} \to y = (x - 1)^2 = x^2 - 2x + 1$$

$$V = \pi \int_1^3 [(x - 1)^2]^2 dx = \pi \int_1^3 (x - 1)^4\, dx = \pi \left[\frac{1}{5}(x - 1)^5\right]_1^3 = \frac{\pi}{5}[32 - 0] = \frac{32\pi}{5}$$

We will next time also talk about surface area of a solid of revolution.