4/19/2022

Chapter 6: Applications of Integration
6.1 Area Between Curves
6.2/6.3 Volumes of Revolution (?)

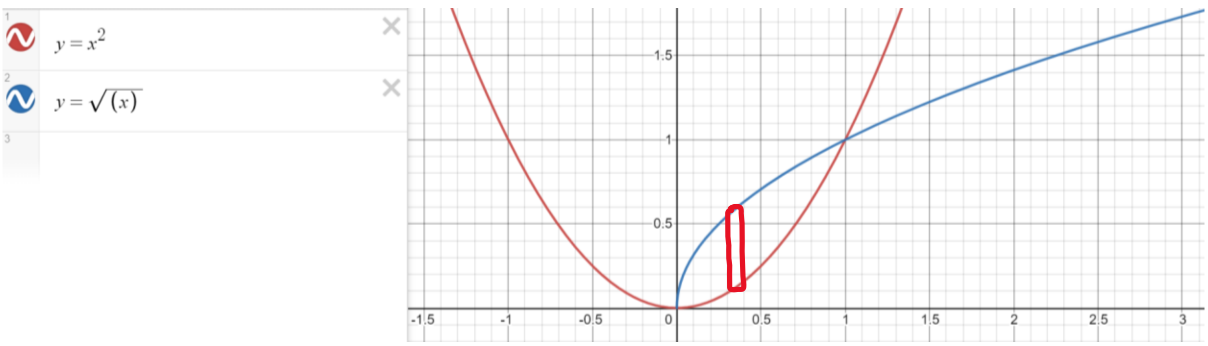
Area between curves.

Finding the area between curves.
In the previous area problems, we were finding the area between a curve and the x-axis. We can think of the x-axis as a function. $f(x) = 0$.
If we want to find the area under a curve (between the curve g(x) and the x-axis)

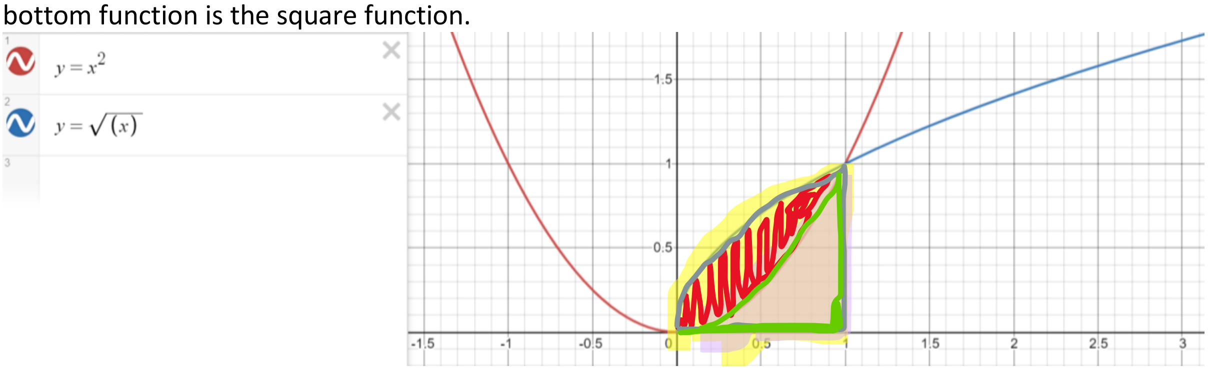
$$\int_a^b [g(x) - f(x)]dx = \int_a^b [g(x) - 0]dx = \int_a^b g(x)dx$$

In the more general case, we use f(x) for some other function not zero.

In the general the top function is the "first" function (g(x)), and the "bottom" function is the second one in the formula.

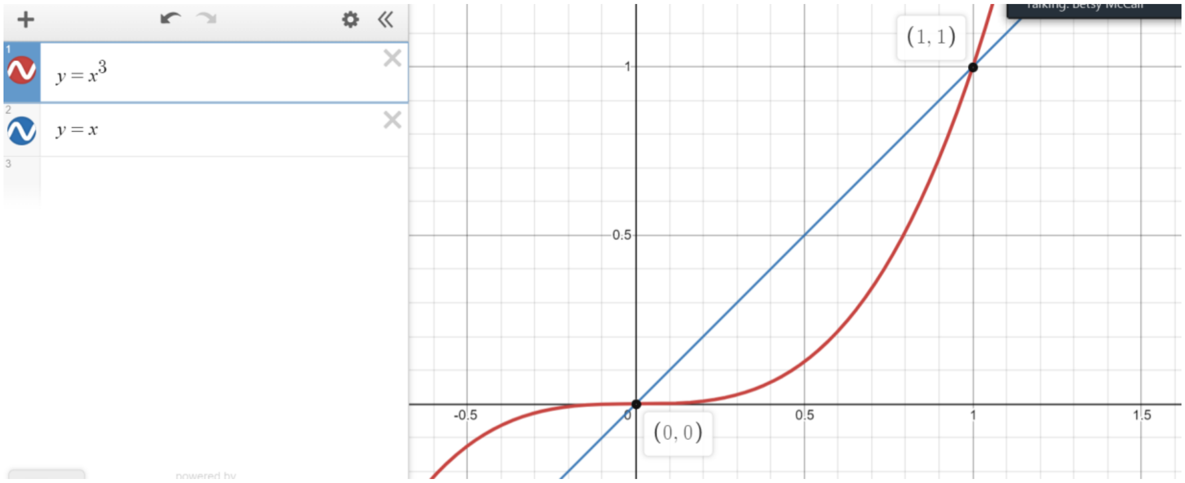


In between these two curves is a region where the square root function is the "top" function, an the bottom function is the square function.



The area under the top curve then subtract off the area under the bottom curve that we don't need.

$$\int_a^b g(x)dx - \int_a^b f(x)dx = \int_a^b [g(x) - f(x)]dx$$



Find the area between $f(x) = x^3$ *and* $g(x) = x$ in the first quadrant.

$$x^3 = x$$
$$x^3 - x = 0$$
$$x(x^2 - 1) = x(x - 1)(x + 1) = 0$$
$$x = -1,0,1$$

I can ignore the $x = -1$ because this isn't in the first quadrant.

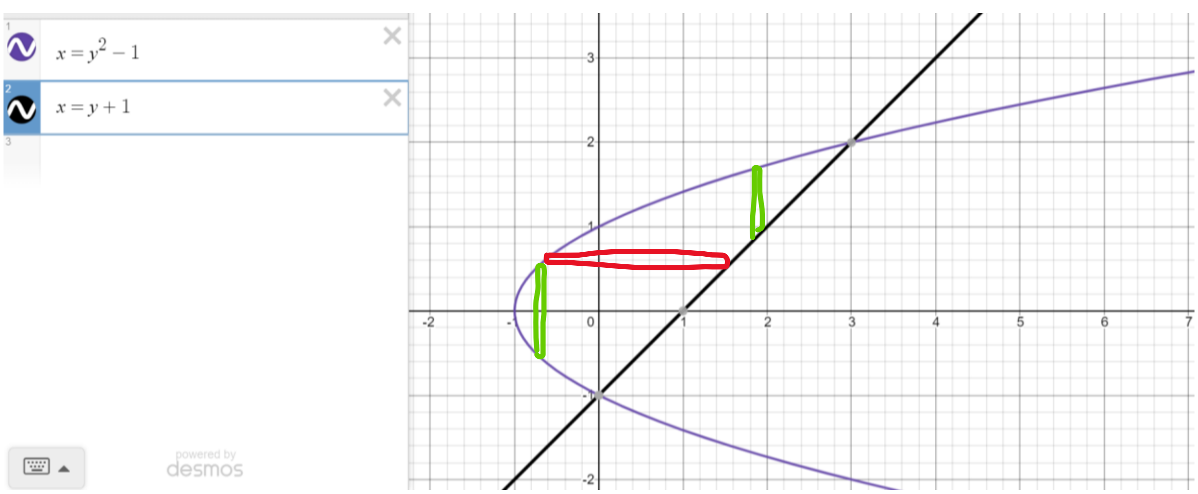
$$A = \int_0^1 x - x^3 dx = \frac{1}{2}x^2 - \frac{1}{4}x^4\Big|_0^1 = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$$

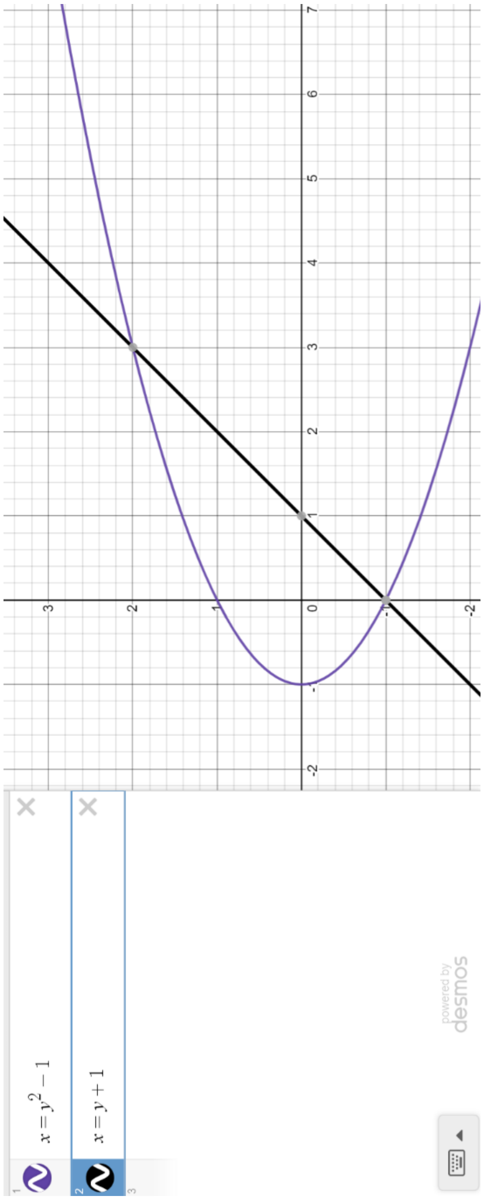
In the third quadrant, the cubic is above the linear function

$$A = \int_{-1}^0 x^3 - x\,dx = \frac{1}{4}x^4 - \frac{1}{2}x^2\Big|_{-1}^0 = 0 - \left(\frac{1}{4} - \frac{1}{2}\right) = -\left(-\frac{1}{4}\right) = \frac{1}{4}$$

So, if you just switch the order of the functions, then the value will differ from the true value by a sign. But, if the order of the functions change in the region of integration, you have to split them at the switch.

Horizontal rectangles.
Use with functions where x is solved in terms of y. not $y = f(x)$ but $x = f(y)$.





Right=top and left=bottom

The larger value in x (or y) is the top (right) function, and the smaller value in x (or y) is the bottom (left) function.

$$\int_a^b top - bottom\, dx$$

$$\int_c^d right - left\, dy$$

The intersections you will need to use the y-coordinates of the points not the x (since we are integrating in y).

Another possible trick you can use to get oriented is to switch all the variables. Replace x with y, and y with x. That would include any constants in the problem. So, if you have an equation for a boundary x=2, that would also have to become y=2.

Traditional orientation:

$$\int_{-1}^{0} \sqrt{x+1} - \left(-\sqrt{x+1}\right)dx + \int_0^3 \sqrt{x+1} - (x-1)dx$$

Solve for $x = y^2 - 1$ for y.

$$x + 1 = y^2$$
$$y = \pm\sqrt{x+1}$$

Solve $x = y + 1$ for y.

$$y = x - 1$$

But, if use the horizontal orientation:

$$\int_{-1}^{2} (y+1) - (y^2 - 1)dy$$

$$\int_{-1}^{0} 2\sqrt{x+1}dx + \int_0^3 \sqrt{x+1} - x + 1dx$$

$$2(\frac{2}{3}(x+1)^{\frac{3}{2}}\Big|_{-1}^{0} + \frac{2}{3}(x+1)^{\frac{3}{2}} - \frac{1}{2}x^2 + x\Big|_0^3 = 2\left(\frac{2}{3} - 0\right) + \frac{2}{3}(4)^{\frac{3}{2}} - \frac{9}{2} + 3 - \frac{2}{3} = \frac{4}{3} - \frac{2}{3} + 3 - \frac{9}{2} + \frac{16}{3}$$
$$= \frac{9}{2}$$

$$\int_{-1}^{2} y - y^2 + 2dy = \frac{y^2}{2} - \frac{y^3}{3} + 2x\Big|_{-1}^{2} = 2 - \frac{8}{3} + 4 - \left(\frac{1}{2} + \frac{1}{3} - 2\right) = \frac{9}{2}$$