3/31/2022

5.2 The Definite Integral
5.3 The Fundamental Theorem of Calculus
5.4 Integration formulas and net change

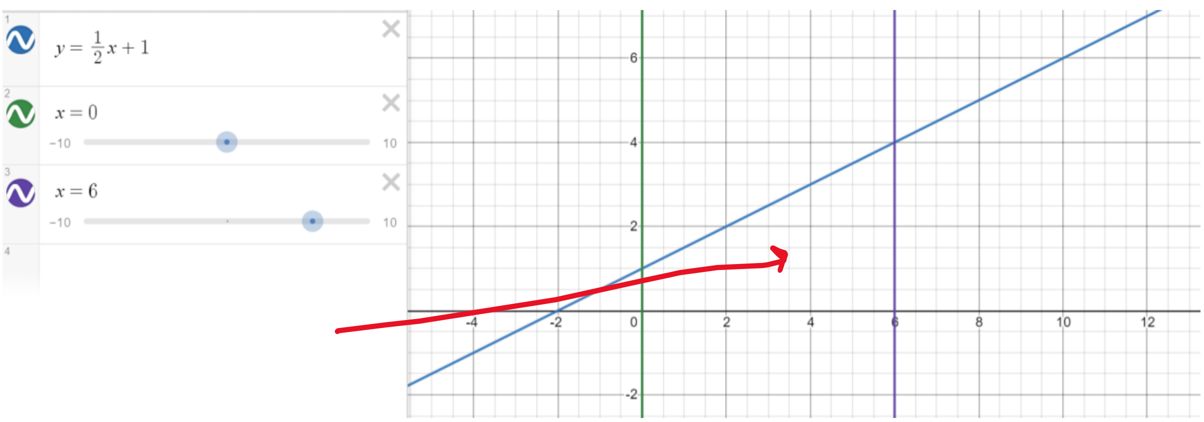
Formal definition of the definite integral and the Fundamental theorem of integrals:

$$\int_a^b f(x)dx = A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

Area between the function and the x-axis.
The only restriction on Δx is that it gets smaller as n gets bigger.

The definite integral is a representation of the area. The definite integral can be given in the problem, and then then tell you to approximate it with a certain number of rectangles, or using Riemann sums (name of the summation method from 5.1). And sometimes they will have us find the area geometrically rather than through integration.

Look at either a triangle or trapezoid example.
Look at a circle (semicircle) example.

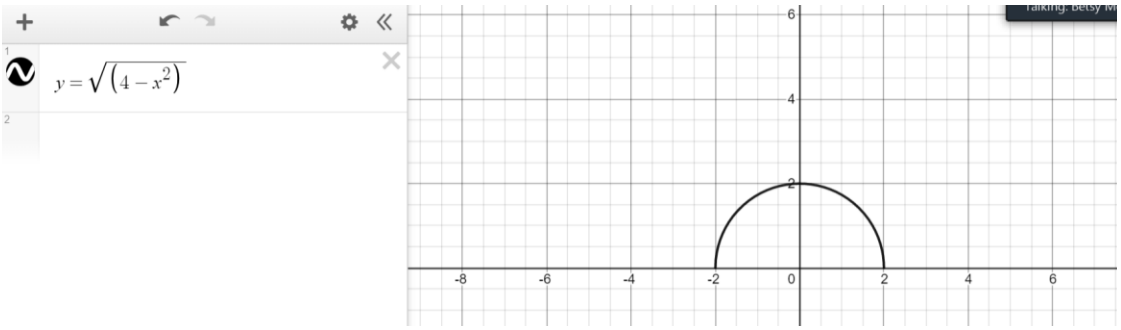


How would we express this area as an integral?

$$A = \int_0^6 \frac{1}{2}x + 1\, dx$$

This region is a trapezoid. $A = \frac{1}{2}(b_1 + b_2)h$

$$A = \frac{1}{2}(1+4)(6) = (3)(5) = 15$$



Use geometry to find the area under this curve since it is a semicircle and we can't find the antiderivative of this function (yet).

$$A = \int_{-2}^{2} \sqrt{4 - x^2}\, dx$$

$$A = \frac{1}{2}\pi r^2 = \frac{1}{2}\pi(2)^2 = 2\pi$$

You may be asked to find the area of a region geometrically.
For example, the absolute value, the graph is a V-shaped graph.

$$A = \int_{-2}^{5} |x|\, dx$$

Find the area of the two small triangles.

Properties:

Definite integrals have signed areas: areas that appear under the x-axis are negative when they come out of the integral. And if the function is above the x-axis, then the area will come out positive.

$$\int_{a}^{a} f(x)dx = 0$$

This makes sense because the width of any rectangle in this interval of zero width is also going to be zero width.

$$\int_{a}^{b} f(x)dx = -\int_{b}^{a} f(x)dx$$

Properties of definite and indefinite integrals are the same

$$\int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$$

$$\int_a^b cf(x)dx = c\int_a^b f(x)dx$$

Splitting a definite integral into pieces:

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

This is more conceptually clear if c is between a and b, but it works even if c is bigger than b.

Use properties examples.

$$\int_1^5 f(x)dx = -3, \int_2^5 f(x)dx = 4$$

What is the value of the integral $\int_1^2 f(x)dx$?

$$\int_1^5 f(x)dx = \int_1^2 f(x)dx + \int_2^5 f(x)dx$$

$$-3 = \int_1^2 f(x)dx + 4$$

$$\int_1^2 f(x)dx = -7$$

If we have a relationship between two functions:
If $f(x) \geq 0,$ then $\int_a^b f(x)dx \geq 0$ as long as $a \leq b$

And if $f(x) \geq g(x)$ then $\int_a^b f(x)dx \geq \int_a^b g(x)dx$

The average value for a function.

$$\bar{f} = \frac{1}{b-a}\int_a^b f(x)dx$$

The average height of the function on the interval [a,b].

(related to the mean value theorem for integrals—maybe we'll see it later)

The (First) Fundamental Theorem of Calculus

$$\int_a^b f(x)dx = F(b) - F(a)$$

The area is related the antiderivatives.

Find the area of the trapezoid using calculus.

$$A = \int_0^6 \frac{1}{2}x + 1\,dx$$

$$\int_0^6 \frac{1}{2}x + 1\,dx = \frac{1}{2}\left(\frac{x^2}{2}\right) + x\Big|_0^6 = \frac{6^2}{4} + 6 - (0 + 0) = 9 + 6 = 15$$

This is same value we got from doing the geometry.

Ex. Find the area under the curve $\int_1^4 e^x + xdx$

$$\int_1^4 e^x + xdx = e^x + \frac{x^2}{2}\Big|_1^4 = e^4 + 8 - \left(e^1 + \frac{1}{2}\right) = e^4 - e + \frac{15}{2} \approx 59.379868\ldots$$

The (Second) Fundamental Theorem of Calculus

$$g(x) = \int_a^x f(t)dt$$

An accumulation function: as the length of the interval increases, how much “area” is accumulated.
If $f(t)$ is a rate function, like velocity, then g(x) is the total accumulated distance traveled.
If $f(t)$ is a rate function like interest earned, then the g(x) is total accumulated interest

What if we want to take the derivative of this function?

$$g'(x) = \frac{d}{dx}\left[\int_a^x f(t)dt\right] = f(x)$$

Where this becomes more complicated is if the upper limit is a function of x, and not just x.

$$g(x) = \int_a^{h(x)} f(t)dt$$

$$g'(x) = f\big(h(x)\big) \cdot h'(x)$$

$$g(x) = \int_1^x e^{t^2}dt$$

$$g'(x) = e^{x^2}$$

$$g(x) = \int_2^{\sin x} e^{t^2} dt$$

$$g'(x) = e^{\sin^2 x} \cos\ x$$

If both limits have x in them, split the integral and flip one.

$$g(x) = \int_x^{x^2} \tan t\, dt = \int_x^0 \tan t\, dt + \int_0^{x^2} \tan t\, dt = -\int_0^x \tan t\, dt + \int_0^{x^2} \tan t\, dt$$

$$g'(x) = -\tan x + \tan x^2 \cdot 2x$$

5.4 is basic integration formulas : essentially a rehash of 4.10

We can rearrange the Fundament theorem expression

$$\int_a^b f(x)dx = F(b) - F(a)$$

$$F(a) + \int_a^b f(x)dx = F(b)$$

This the “Net Change Theorem”.

Properties of even and odd function integrals.

$$\int_{-a}^{a} f(x)dx$$

If $f(x)$ is even, then $\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx$ (the area to the right of 0 is the same as the area to the left of zero)

If $f(x)$ is off, then $\int_{-a}^{a} f(x)dx = 0$ (the area to one side is negative, and the area to the other side is positive, and by symmetry the same size)

$$\int_{-1}^{1} x^5 + x^4 - x^3 + 1dx = \int_{-1}^{1} x^5 - x^3 dx + \int_{-1}^{1} x^4 + 1dx = 2\int_0^1 x^4 + 1dx = \frac{x^5}{5} + x\Big|_0^1 = \frac{1}{5} + 1 = \frac{6}{5}$$

(I forgot to turn the recording back on for this last example! Sorry!, but the idea is just to use the properties to make your life easier, rather than do all the terms and plug in a lot of non-zero values and deal with all the signs and cancelling, you can cancel early and do less work.)