3/3/2022

4.1 Related Rates (one more example)
4.2 Linear Approximations/Differentials
4.3 Maxima and Minima

Example 9 Suppose that we have two resistors connected in parallel with resistances R_1 and R_2 measured in ohms (Ω). The total resistance, R, is then given by,

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$$

Suppose that R_1 is increasing at a rate of 0.4 Ω/min and R_2 is decreasing at a rate of 0.7Ω/min. At what rate is R changing when $R_1 = 80\,\Omega$ and $R_2 = 105\,\Omega$?

(from Paul's Online Math Notes)

Take the derivative of the function with all variables in place first.

$$R^{-1} = R_1^{-1} + R_2^{-1}$$

$$-R^{-2}\frac{dR}{dt} = -1R_1^{-2}\frac{dR_1}{dt} - 1R_2^{-2}\frac{dR_2}{dt}$$

$$\frac{1}{R} = \frac{1}{80} + \frac{1}{105} = 0.22023 \ldots = \frac{37}{1680}$$

$$R = \frac{1680}{37}$$

$$-\left(\frac{1680}{37}\right)^{-2}\frac{dR}{dt} = -(80)^{-2}\,(0.4) - (105)^{-2}\,(-0.7)$$

$$\frac{dR}{dt} = -\left(\frac{1680}{37}\right)^{2}\left[-\frac{0.4}{80^2} + \frac{0.7}{105^2}\right] = -0.002045 \ldots = -\frac{14}{6845}$$

In Ohms per minute

Linear Approximations and Differentials

A tangent line is essentially a linear approximation to the curve near the point.

$$f(x) \approx f(a) + f'(a)(x-a)$$

$$L(x) = f(a) + f'(a)(x-a)$$

These are the same. $f'(a)$ is the slope of the tangent line at a. $f(a)$ is the value of the point the curve and the tangent share. $x - a$ is the difference between the point x where we are doing the approximation, and a where we are centering our estimate. $(x-a)$ must generally be small to be accurate.

$$y = f(a)$$

$$\Delta y = f'(a)\Delta x = (a - x)\ \ or\ (x - a)$$

$$f(x + \Delta x) \approx y + \Delta y = f(a) + f'(a)\Delta x$$

$$f(x + \mathrm{dx}) \approx y + \mathrm{dy} = f(a) + f'(a)\mathrm{dx}$$

Δx is the change in x (any size change)
dx is an infinitesimal change in x (meant to be small)

Find the linear approximation to the graph of $y = (x - 1)^4$ at $x = 0$.

$$a = 0$$
$$f(a) = 1$$
$$f'(x) = 4(x - 1)^3(1) = 4(x - 1)^3$$
$$f'(a) = -4$$

$$L(x) = 1 + (-4)(x - 0)$$
$$L(x) = -4x + 1$$

The approximation will only be "good" near x=0.

Differentials
Estimate the value of $f(3.1)$ for $f(x) = x^2 + 2x$ using differentials.

$$f'(x) = 2x + 2$$
We want to build our estimate from a nice, easy to evaluate point. So, what is a? $a = 3$, $\Delta x = dx = 0.1$

$$dy = f'(a)dx = (2(3) + 2)(0.1) = 0.8$$

Estimate for the new y value is old y value plus the change:
$$f(3.1) \approx f(3) + dy = [3^2 + 2(3)] + 0.8 = 15.8$$

True value of $f(3.1) = 15.81$

The function could be "hidden".
Estimate the value of $\sqrt[5]{32.5}$ using differentials or linear approximation.

$$f(x) = \sqrt[5]{x}$$
$a = 32$, the closest 5th root we can find
Then dx is the difference, $dx = 0.5$

$$f'(x) = \frac{1}{5}x^{-\frac{4}{5}} = \frac{1}{5\sqrt[5]{x^4}}$$

$$f'(32) = \frac{1}{5\left(\sqrt[5]{32}\right)^4} = \frac{1}{5(2^4)} = \frac{1}{5(16)} = \frac{1}{80}$$

$$dy = \frac{1}{80}(0.5) = \frac{1}{160}$$

$$\sqrt[5]{32.5} \approx \sqrt[5]{32} + \frac{1}{160} = 2 + \frac{1}{160} = 2.00625$$

True value : 2.0062113…

Maxima and Minima

(Relative) Maximum is a point that is larger than any other nearby point, the plural is maxima
(Relative) Minimum is a point that is smaller than any other nearby point, the plural is minima
Absolute maximum is a point that is larger than any other point on the function (y)
Absolute minimum is a point that is smaller than any other point on the function (y)
Not all functions have maxima or minima (linear functions have neither relative or absolute minima or maxima)

Extremum (plural is extrema) is either a maximum or a minimum (an extreme value, or an extreme point).

Relative extremum or local extremum (these are the same)
Absolute extremum or global extremum (these are the same)

Relative extrema occur at critical points
Critical points are places there the derivative is either zero or undefined
Not every critical point is an extremum, but all extrema occur at critical points.

Turning point in a curve, has a derivative which is zero (the tangent line is horizontal)
A cusp is a point where the curve is continuous but makes a sharp turn, here, the derivative is undefined
But, some places where the derivative is zero are not turning points, and some places where the derivative is undefined, the original function is also not defined (vertical asymptotes), some places the curve is continuous but you have a vertical tangent line.

Steps to finding extrema

1. Take the derivative
2. Set the derivative equal to zero.
3. Do algebra to find any values where this is so.
4. Find any places where the derivative is undefined.
5. Check (either a sign chart—first derivative test—or the sign of the second derivative—the second derivative test) to determine whether it is a maximum, minimum or neither (or we might find out that we can’t tell).

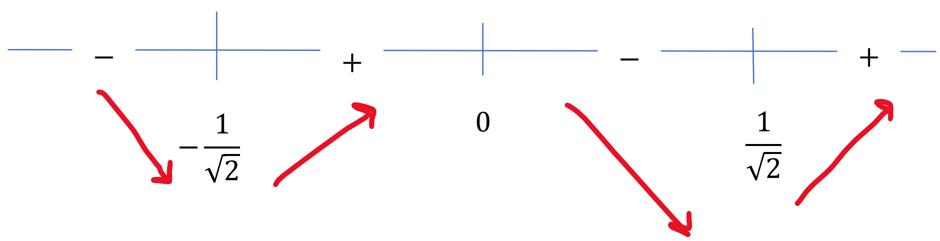
$$y = x^2(x-1)(x+1) = x^2(x^2-1) = x^4 - x^2$$

$$f'(x) = 4x^3 - 2x$$
$$4x^3 - 2x = 0$$

$$2x(2x^2 - 1) = 0$$

$$2x = 0, x = 0$$
$$2x^2 - 1 = 0$$
$$2x^2 = 1$$
$$x^2 = \frac{1}{2}$$
$$x = \pm\frac{1}{\sqrt{2}}$$

Our critical points are at $x = 0, x = \frac{1}{\sqrt{2}}, x = -\frac{1}{\sqrt{2}}$



The first derivative test is built on a sign chart. What is the sign of the derivative in between these critical points.
Test points: -1,-1/2, ½, 1. Plug into the derivative.

$$2x(2x^2 - 1)$$

$$x = -1, (-)(+) = (-)$$
$$x = -\frac{1}{2}, (-)(-) = (+)$$
$$x = \frac{1}{2}, (+)(-) = (-)$$
$$x = 1, (+)(+) = (+)$$

If your sign chart has a negative followed by a positive, this is a minimum
If your sign chart has a positive followed by a negative, this is a maximum.

If neither, then it is neither a minimum nor a maximum.

(this assumes that all these critical are defined on the original function.)

You can't have a maximum if the function is not defined. You should include places where the derivative is not defined but the function is (cusps) in the sign chart.

Next time we'll pick up with the second derivative test and talk about concavity and inflection points. (and we'll do another example).