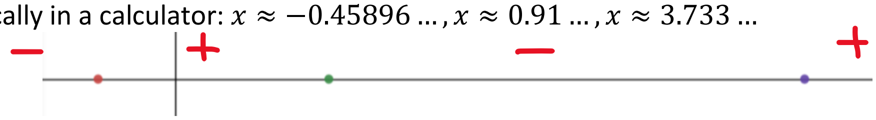
3/22/2022

4.6 Some more examples
4.7 Applied Optimization Problems
4.8 L'Hôpital's Rule

Examples.
Sketch the graph of $f(x) = e^x - x^3$.

$$f'(x) = e^x - 3x^2$$
$$e^x = 3x^2$$

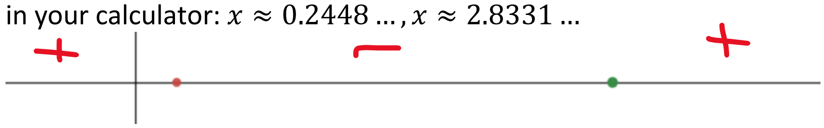
Solve numerically in a calculator: $x \approx -0.45896\ldots, x \approx 0.91\ldots, x \approx 3.733\ldots$



Test points: -1, 0, 1, 4

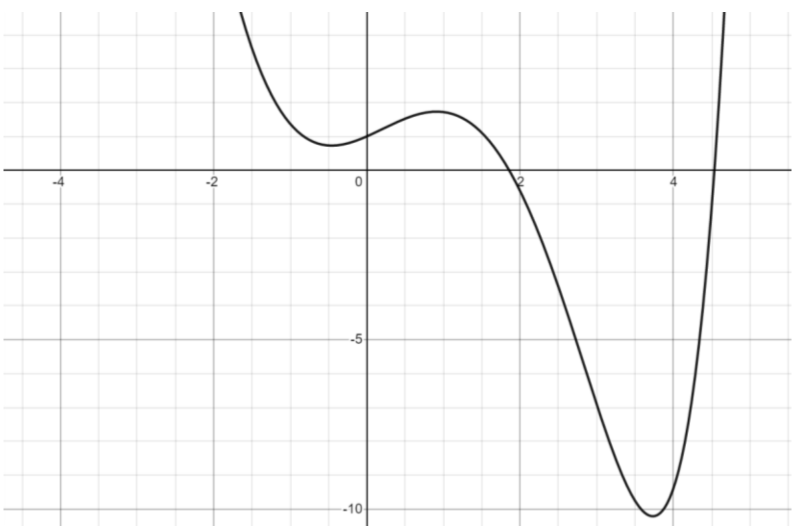
$$f''(x) = e^x - 6x$$
$$e^x - 6x = 0$$
$$e^x = 6x$$

Solve numerically in your calculator: $x \approx 0.2448\ldots, x \approx 2.8331\ldots$



Test points: 0, 1, 3

x-intercepts have to be solved for numerically. There is a y-intercept (0,1).



Sketch the graph of $f(x) = \frac{x^2+x-2}{x^2-3x-4}$.

$$f'(x) = \frac{(2x+1)(x^2-3x-4) - (2x-3)(x^2+x-2)}{(x^2-3x-4)^2}$$

$$\frac{(2x^3 - 6x^2 - 8x + x^2 - 3x - 4) - (2x^3 + 2x^2 - 4x - 3x^2 - 3x + 6)}{(x^2 - 3x - 4)^2} = 0$$

$$2x^3 - 5x^2 - 11x - 4 - (2x^3 - x^2 - 7x + 6) = 2x^3 - 5x^2 - 11x - 4 - 2x^3 + x^2 + 7x - 6 = 0$$

$$-4x^2 - 4x - 10 = -2(2x^2 - 2x + 5) = 0$$

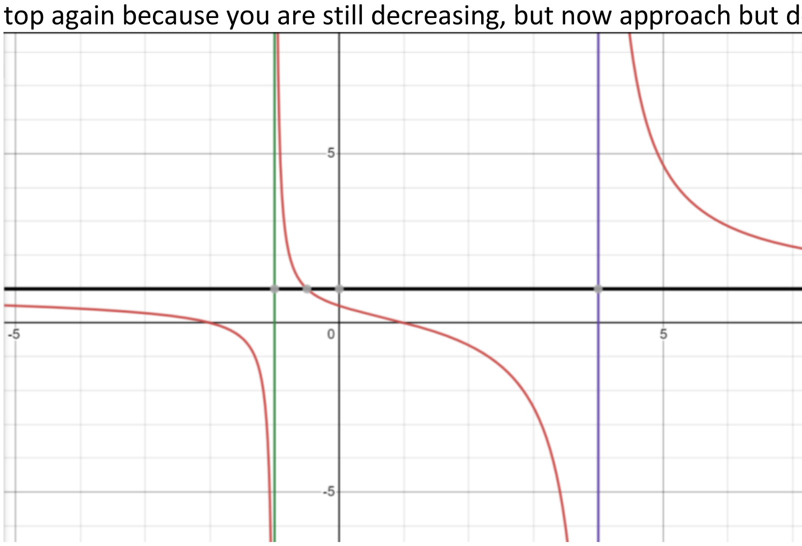
This derivative never changes sign. The quadratic is always positive, so the whole derivative is always negative, so it is always decreasing.

Don't worry about the second derivative here. It's really messy. From algebra, find the asymptotes.

Vertical Asymptotes when $x^2 - 3x - 4 = (x - 4)(x + 1) = 0$. So when $x = -1, x = 4$.
Horizontal asymptote when $\frac{x^2}{x^2} = 1$, so when $y = 1$.

The y-intercept is $\left(0, \frac{1}{2}\right)$. The x-intercepts are when $x^2 + x - 2 = (x + 2)(x - 1) = 0, x = -2, x = 1$

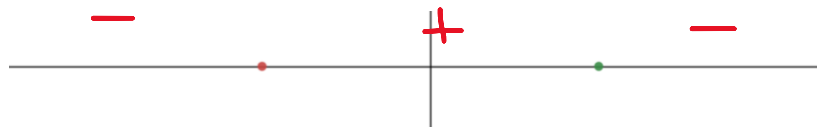
Sketch the graph start with being below the asymptote and decreasing on the left passing through $x = -2$. When you hit the asymptote, you still have to be decreasing, so start at the top on the other side and keep decreasing. Through the other intercepts. Then when you hit the next asymptote, start at the top again because you are still decreasing, but now approach but do not cross the horizontal asymptote.



Sketch the graph of $f(x) = \frac{3x}{x^2+1}$.

$$f'(x) = \frac{3(x^2 + 1) - 2x(3x)}{(x^2 + 1)^2} = \frac{-3x^2 + 3}{(x^2 + 1)^2}$$

$$3x^2 + 3 - 6x^2 = 0$$
$$-3(x^2 - 1) = 0$$
$$x = \pm 1$$



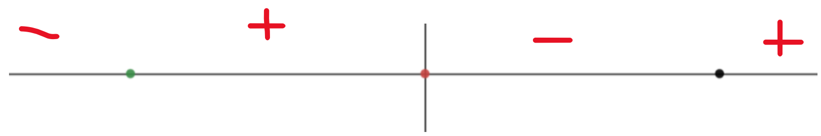
Test points: -2, 0, 2

$$f''(x) = \frac{\left((-6x)(x^2+1)^2 - 2(x^2+1)(2x)(-3x^2+3)\right)}{(x^2+1)^4}$$

$$\frac{(x^2+1)[-6x(x^2+1) - 4x(-3x^2+3)]}{(x^2+1)^4} = \frac{-6x(x^2+1) - 4x(-3x^2+3)}{(x^2+1)^3}$$

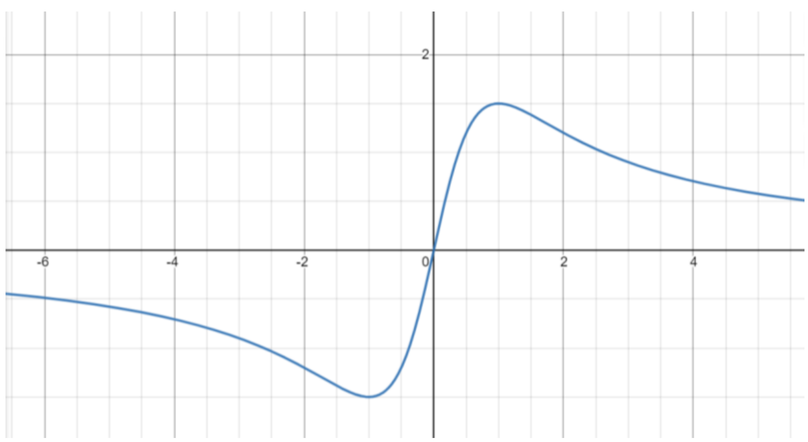
$$-6x(x^2+1) - 4x(-3x^2+3) = -6x^3 - 6x + 12x^3 - 12x = 6x^3 - 18x = 0$$
$$6x(x^2-6) = 0$$

$$x = 0, x = \pm\sqrt{6}$$



Test points: -3,-1,1,3

There are no vertical asymptotes since $x^2 + 1$ is never zero. There is a horizontal asymptote at $y = 0$. There is an intercept at $(0,0)$.



Sketch the graph $f(x) = x^{2/3} - x^{2/5}$.

$$f'(x) = \frac{2}{3}x^{-\frac{1}{3}} - \frac{2}{5}x^{-\frac{3}{5}} = \frac{2}{3x^{\frac{1}{3}}} - \frac{2}{5x^{\frac{3}{5}}} = \frac{2}{3x^{\frac{5}{15}}} - \frac{2}{5x^{\frac{9}{15}}} = \frac{2}{x^{\frac{5}{15}}}\left(\frac{1}{3} - \frac{1}{5x^{\frac{4}{15}}}\right)$$

The factor $\frac{2}{x^{\frac{1}{3}}}$ is undefined when $x = 0$.

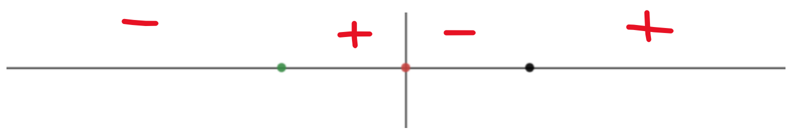
$$\frac{1}{3} - \frac{1}{5x^{\frac{4}{15}}} = 0$$

$$\frac{1}{3} = \frac{1}{5x^{\frac{4}{15}}}$$

$$5x^{\frac{4}{15}} = 3$$

$$x^{\frac{4}{15}} = \frac{3}{5}$$

$$x = \pm \sqrt[4]{\left(\frac{3}{5}\right)^{15}} \approx \pm 0.14725 \ldots$$



Test points: -1, -0.1, 0.1, 1

$$f''(x) = -\frac{2}{9}x^{-\frac{4}{3}} + \frac{6}{25}x^{-\frac{8}{5}} = -\frac{2}{9x^{\frac{4}{3}}} + \frac{6}{25x^{\frac{8}{5}}} = -\frac{2}{9x^{\frac{20}{15}}} + \frac{6}{25x^{\frac{24}{15}}} = \frac{2}{x^{\frac{20}{15}}}\left(-\frac{1}{9} + \frac{3}{25x^{\frac{4}{15}}}\right)$$

The factor $\frac{2}{x^{\frac{4}{3}}}$ is undefined when $x = 0$.

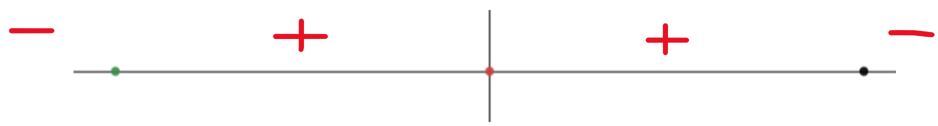
$$-\frac{1}{9} + \frac{3}{25x^{\frac{4}{15}}} = 0$$

$$\frac{1}{9} = \frac{3}{25x^{\frac{4}{15}}}$$

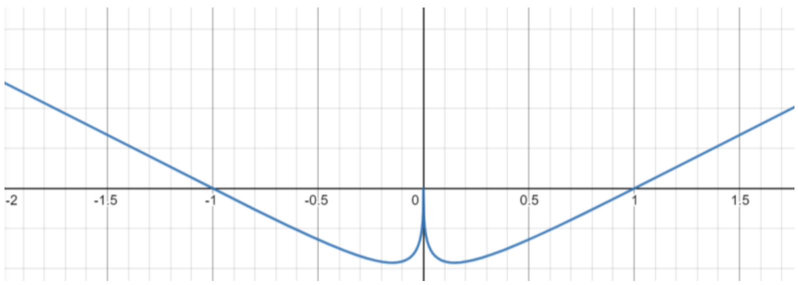
$$25x^{\frac{4}{15}} = 27$$
$$x^{\frac{4}{15}} = \frac{27}{25}$$

$$x = \pm \sqrt[4]{\left(\frac{27}{25}\right)^{15}} \approx \pm 1.33456 \ldots$$

Test points: -2,-1,1,2



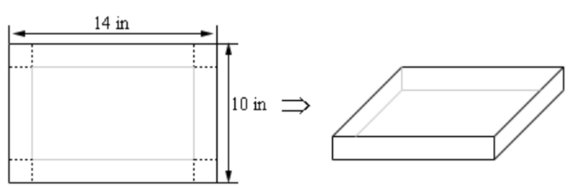
There is an intercept at $(0,0)$, where there is a cusp (the derivative is not defined at this point), and x-intercepts at $x = \pm 1$.



Optimization problems.

Application problems: set up the equation, and then do all the optimization stuff we've done before.

Example 5 We have a piece of cardboard that is 14 inches by 10 inches and we're going to cut out the corners as shown below and fold up the sides to form a box, also shown below. Determine the height of the box that will give a maximum volume.



The sides of the square we cut out is x, and that becomes the heigh of the box. $V = lwh = (14 - 2x)(10 - 2x)x = x(140 - 28x - 20x + 4x^2) = 4x^3 - 48x^2 + 140x$.

To find the maximum volume, take the derivative and set it equal to zero to find the critical points.

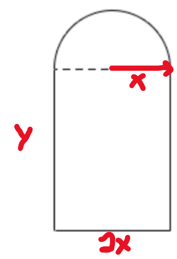
$$V'(x) = 12x^2 - 96x + 140 = 4(3x^2 - 24x + 35)$$

$$x = \frac{24 \pm \sqrt{24^2 - 4(3)(35)}}{2(3)} = \frac{24 \pm \sqrt{156}}{6} \approx 6.08\ldots, 1.918\ldots$$

Since the width is $10 - 2x$ the biggest x can be and still have a physical width is $x = 5$, so I can discard the 6.08 value (this will be the function minimum since it creates a negative volume). So, the maximum occurs at the other critical point: to get the maximum volume, plug that back into the volume equation.

The maximum volume is around 473 cubic inches.

Example 1 A window is being built and the bottom is a rectangle and the top is a semicircle. If there is 12 meters of framing materials what must the dimensions of the window be to let in the most light?



Decide are we going to use x as the radius of the semicircle, or x as the width of the rectangle (=diameter of the semicircle).
We want to maximize the area. We need to find the total area. The fact that there is 12 meters of framing material will allow us to relate the x- and y-variables so our optimization problem has only one variable.

The arc of the semicircle is half the circumference of the circle. $C = 2\pi r$. For the arc $L = \pi x$.
The sides of the rectangle for the frame are $R = 2y + 2x$
Total frame: $L + R = \pi x + 2x + 2y = 12$

Area of the semicircle is half the circle: $A = \pi r^2$, $A_s = \frac{1}{2}\pi x^2$
Area of the rectangle is $A = lw$. $A_r = 2xy$

Total area: $A = \frac{1}{2}\pi x^2 + 2xy$

Solve framing material equation for y.

$$2y = 12 - \pi x - 2x$$

$$y = 6 - \frac{\pi}{2}x - x$$

Area in terms of x alone:

$$A = \frac{1}{2}\pi x^2 + 2x\left(6 - \frac{\pi}{2}x - x\right)$$

$$A = \frac{1}{2}\pi x^2 + 12x - \pi x^2 - 2x^2 = 12x - \frac{1}{2}\pi x^2 - 2x^2$$

Now, take the derivative.

$$A' = 12 - \pi x - 4x = 0$$

$$12 = x(\pi + 4)$$

$$x = \frac{12}{\pi + 4} \approx 1.68 \ldots$$

x is the radius of the semicircle.
The width of the window is $2x \approx 3.36$
The height of the window is $y = 6 - \frac{\pi}{2}x - x \approx 1.68$

Squares maximize area.

Example 3 Determine the point(s) on $y = x^2 + 1$ that are closest to $(0, 2)$.

$$D = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

Point 1: $(x, y) = (x, x^2 + 1)$
Point 2: (0,2)

$$D(x) = \sqrt{(x-0)^2 + (x^2+1-2)^2} = \sqrt{x^2 + (x^2-1)^2} = \sqrt{x^2 + x^4 - 2x^2 + 1} = \sqrt{x^4 - x^2 + 1}$$

Closest is minimizing the distance

$$D = (x^4 - x^2 + 1)^{1/2}$$

$$D' = \frac{1}{2}(x^4 - x^2 + 1)^{-\frac{1}{2}}(4x^3 - 2x) = \frac{4x^3 - 2x}{2\sqrt{x^4 - x^2 + 1}}$$

Critical points are when the numerator is zero.

$$4x^3 - 2x = 2x(2x^2 - 1) = 0$$

$$x = 0, \pm\frac{1}{\sqrt{2}}$$

Test points: -1, -½, ½ ,1

$$f'(-1) = (-)$$
$$f'\left(-\frac{1}{2}\right) = (+)$$
$$f'\left(\frac{1}{2}\right) = (-)$$
$$f'(1) = (+)$$

$-\frac{1}{\sqrt{2}}$ goes from decreasing to increasing: minimum
0 goes from increasing to decreasing: maximum
$\frac{1}{\sqrt{2}}$ goes from decreasing to increasing: minimum

There are two minima.

L'Hôpital's Rule

A way of finding limits when the form of the limit is indeterminant

Indeterminant Forms—what are they? Which ones does the rule apply to?

$$\frac{0}{0}, \frac{\infty}{\infty}, 0 \cdot \infty, 1^{\infty}, 0^{0}, 0^{-\infty}, etc.$$

The only ones that L'Hôpital's applies to is the first two.

All the others require algebra to turn them into one of the allowable forms.

If the limit is in one of the allowable indeterminant: then the rule to take the derivative of the numerator and the denominator separately, and then reevaluate the limit.

Example.

$$\lim_{x \to 0} \frac{\sin x}{x} = \frac{0}{0}$$

This is indeterminant of an allowable form. Take the derivative of the numerator and the denominator separately. Then re-evaluate.

$$\lim_{x \to 0} \frac{\cos x}{1} = 1$$

$$\lim_{x \to \infty} \frac{x^2 + x - 2}{x^2 - 3x - 4} = \frac{\infty}{\infty}$$

Apply the rule:

$$\lim_{x \to \infty} \frac{2x + 1}{2x - 3} = \frac{\infty}{\infty}$$

Apply the rule again:

$$\lim_{x \to \infty} \frac{2}{2} = 1$$

The horizontal asymptote is $y = 1$.

Tricky ones are other indeterminant form that we have to modify to use the rule.

$$\lim_{x \to 0^+} x \ln x = 0 \cdot -\infty$$

$$\lim_{x \to 0^+} \frac{\ln x}{x^{-1}} = \lim_{x \to 0^+} \frac{\ln x}{\left(\frac{1}{x}\right)} = -\frac{\infty}{\infty}$$

Apply the rule:

$$\lim_{x\to 0^+} \frac{\left(\frac{1}{x}\right)}{\left(-\frac{1}{x^2}\right)} = \lim_{x\to 0^+} \frac{1}{x} \cdot -\frac{x^2}{1} = \lim_{x\to 0^+} -\frac{x}{1} = 0$$

Example with exponents.

$$\lim_{x\to 0}(1+x)^{1/x}$$

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = L$$

Trick here to get it into an allowable form is to take the log.

$$\ln\left(\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x\right) = \ln L$$

$$\lim_{x\to\infty} ln\left(1+\frac{1}{x}\right)^x = \lim_{x\to\infty} x\, ln\left(1+\frac{1}{x}\right) = \infty \cdot 0$$

$$\lim_{x\to\infty} \frac{\ln\left(1+\frac{1}{x}\right)}{x^{-1}} = \frac{0}{0}$$

$$\lim_{x\to\infty} \frac{\frac{1}{1+\frac{1}{x}}\left(-\frac{1}{x^2}\right)}{-\frac{1}{x^2}} = \lim_{x\to\infty} \frac{1}{\left(1+\frac{1}{x}\right)} = \frac{1}{1} = 1$$

$$\ln L = 1$$

$$L = e$$

We will do Newton’s Method