03/17/2022
Happy St. Patty's Day

4.5 Derivatives and the shape of a graph
4.6 Limits at Infinity/Asymptotes

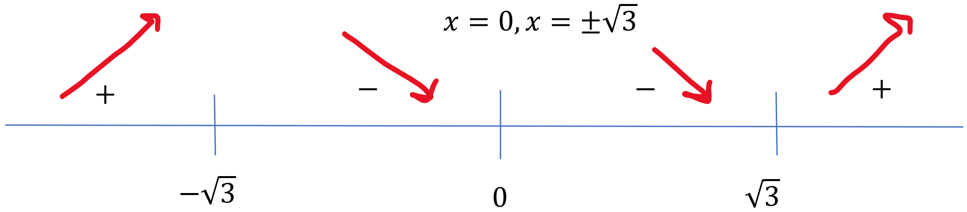
1. Find the first derivative and use it to locate any critical points. Build a sign chart.
2. Find the second derivative and use it to locate any inflection points. Build a sign chart.
3. Combine this information with some algebra (zeros, intercepts, asymptotes, etc.) to build a sketch of the graph.

Example. $f(x) = x^5 - 5x^3$

$$f'(x) = 5x^4 - 15x^2$$
$$5x^2(x^2 - 3) = 5x^2(x - \sqrt{3})(x + \sqrt{3}) = 0$$

Critical points

$$x = 0, x = \pm\sqrt{3}$$



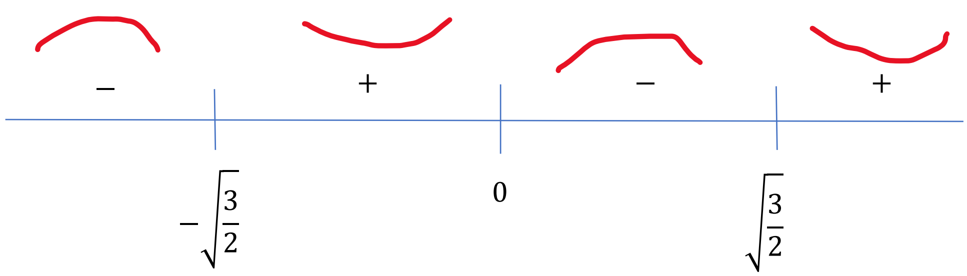
Test points: -2,-1,1,2

$$f'(-2) = (+)(-)(-) = (+)$$
$$f'(-1) = (+)(-)(+) = (-)$$
$$f'(1) = (+)(-)(+) = (-)$$
$$f'(2) = (+)(+)(+) = (+)$$

$$f''(x) = 20x^3 - 30x$$
$$10x(2x^2 - 3) = 0$$

Inflection points

$$x = 0, x = \pm\sqrt{\frac{3}{2}}$$



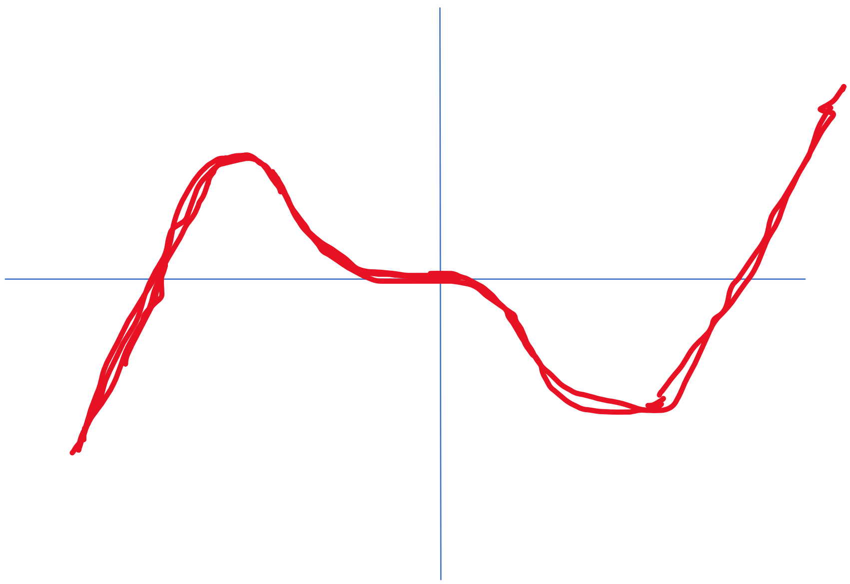
Test points: -2, -1, 1,-2

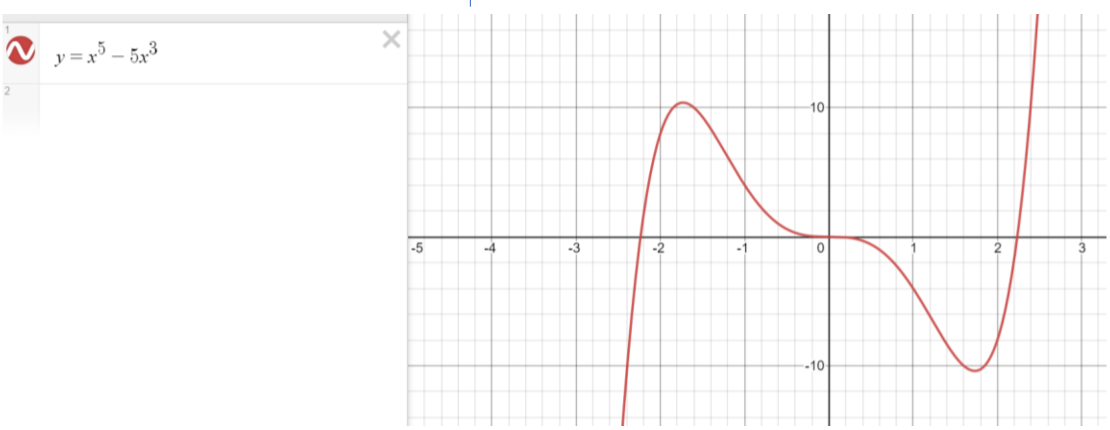
$$f''(-2) = (-)(+) = (-)$$
$$f''(-1) = (-)(-) = (+)$$
$$f''(1) = (+)(-) = (-)$$

$$f''(2) = (+)(+) = (+)$$

No asymptotes.
Zeros? $f(x) = x^5 - 5x^3 = x^3(x^2 - 5)$

$$x = 0, \pm\sqrt{5}$$





Example.

$$f(x) = \sqrt{x}\ln x$$

From algebra: not defined for $x \leq 0$. Not sure what is happening as we get close to 0.
There is an intercept at $x = 1$ because $\ln(1) = 0$.

First derivative:

$$f'(x) = \frac{1}{2\sqrt{x}}\ln x + \sqrt{x}\left(\frac{1}{x}\right) = \frac{\ln x}{2\sqrt{x}} + \frac{1}{\sqrt{x}} = \frac{1}{\sqrt{x}}\left(\frac{1}{2}\ln x + 1\right)$$

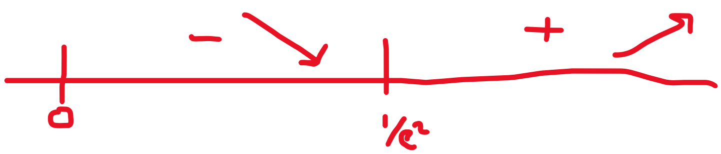
Critical points:

$\frac{1}{\sqrt{x}}$ is not defined at x=0, so this is a critical point… (except the function is not defined at 0, so not an extreme value)

$$\frac{1}{2}\ln x + 1 = 0$$
$$\frac{1}{2}\ln x = -1$$
$$\ln x = -2$$
$$x = e^{-2} = \frac{1}{e^2} \approx 0.1353\ldots$$

(bigger than 0 but less than 1)
Two sections to worry about. There is between 0 and $\frac{1}{e^2}$, and then bigger than $\frac{1}{e^2}$.
Test points are going to be: 0.1, 1

$$f'(0.1) = (+)(-) = (-)$$
$$f'(1) = (+)(+) = (+)$$



Second derivative:

$$f'(x) = \frac{1}{\sqrt{x}}\left(\frac{1}{2}\ln x + 1\right) = x^{-\frac{1}{2}}(\frac{1}{2}\ln x + 1)$$

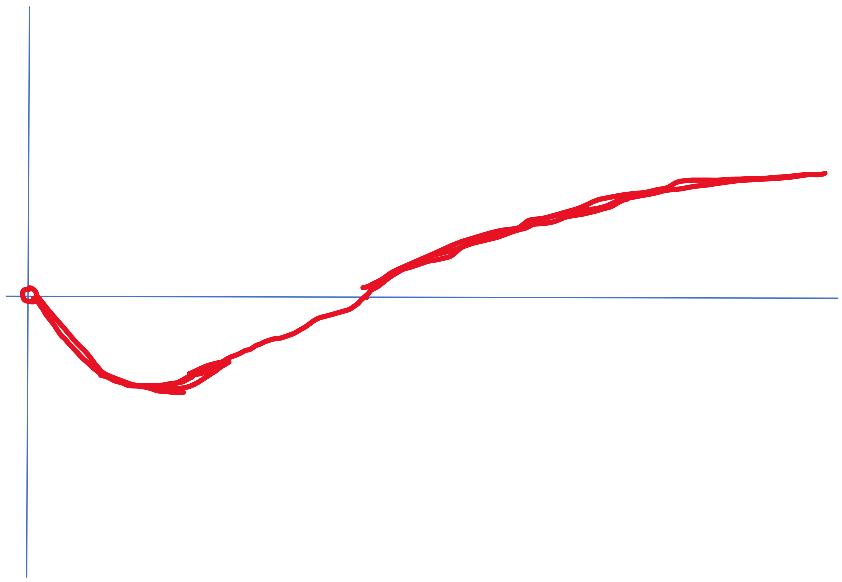
$$f''(x) = -\frac{1}{2}x^{-\frac{3}{2}}\left(\frac{1}{2}\ln x + 1\right) + x^{-\frac{1}{2}}\left(\frac{1}{2x}\right) = -\frac{1}{2\sqrt{x^3}}\left(\frac{1}{2}\ln x + 1\right) + \frac{1}{2\sqrt{x^3}} = \frac{1}{2\sqrt{x^3}}\left(1 - \left(\frac{1}{2}\ln x + 1\right)\right)$$

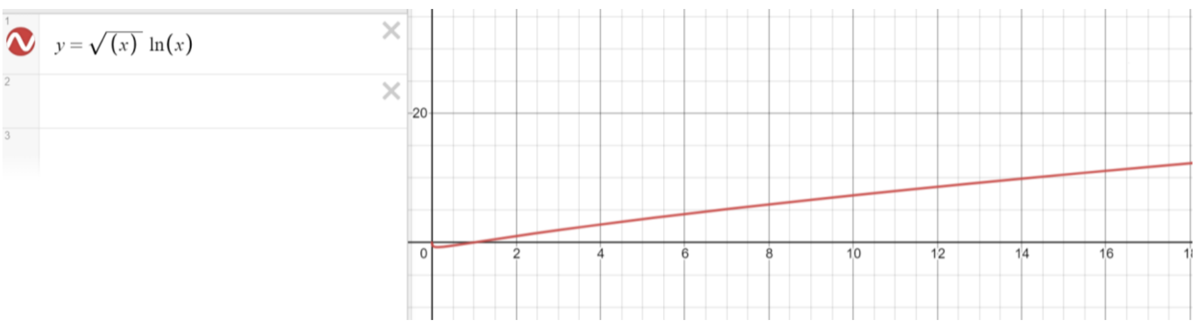
$$f''(x) = \frac{1}{2\sqrt{x^3}}\left(-\frac{1}{2}\ln x\right) = -\frac{\ln x}{4\sqrt{x^3}}$$

Inflection points:
Undefined at x=0 (but the function isn't defined there)
It's equal to zero when numerator is equal to zero. When x=1.

Test points: $\frac{1}{2}$, 2
Only intervals are between 0 and 1, and bigger than 1

$$f''\left(\frac{1}{2}\right) = (-)(-)(+) = (+)$$
$$f''(2) = (-)(+)(+) = (-)$$





(only look at the real graph after trying to sketch it without the graph, just the calculus)

Limits at Infinity and Asymptotes

$$f(x) = 2 + \frac{1}{x} = \frac{2x+1}{x}$$

We might wonder, what is $\lim_{x\to\infty}\left(2+\frac{1}{x}\right) = ?$ and $\lim_{x\to-\infty}\left(2+\frac{1}{x}\right) = ?$

Polynomials will go to infinity (or negative infinity) as x gets very large (or very small—far to the left). Sometimes this is referred to as end behavior.

Rational functions may or may not go to one of the infinities: if the degree of the numerator is larger than the degree of the denominator, then the limits will be infinity or negative infinity.

If the rational function has a numerator and a denominator with the same degree: you get a horizontal asymptote (which is a non-zero constant).

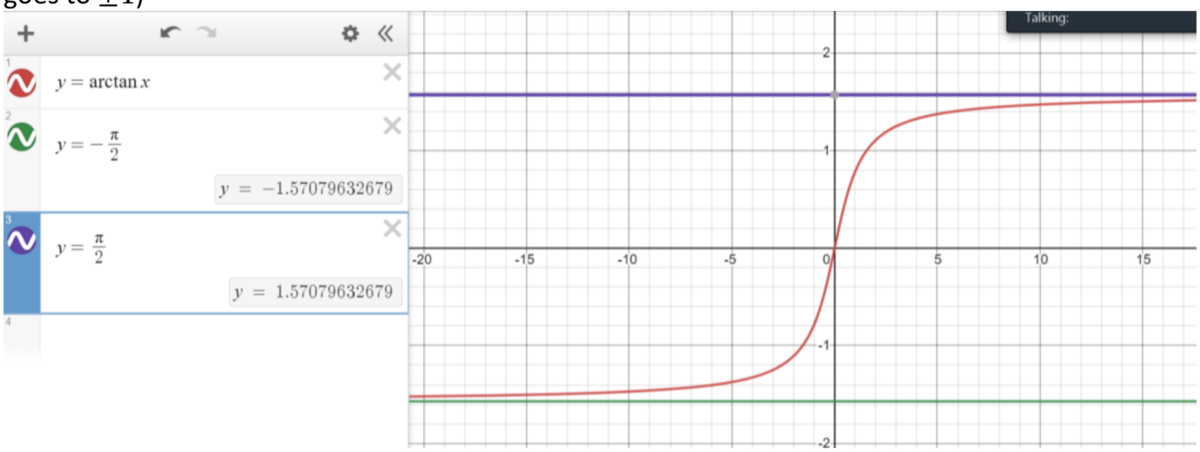
If the rational function has a numerator with a lesser degree than the denominator, then you get a horizontal asymptote, but it is at zero.

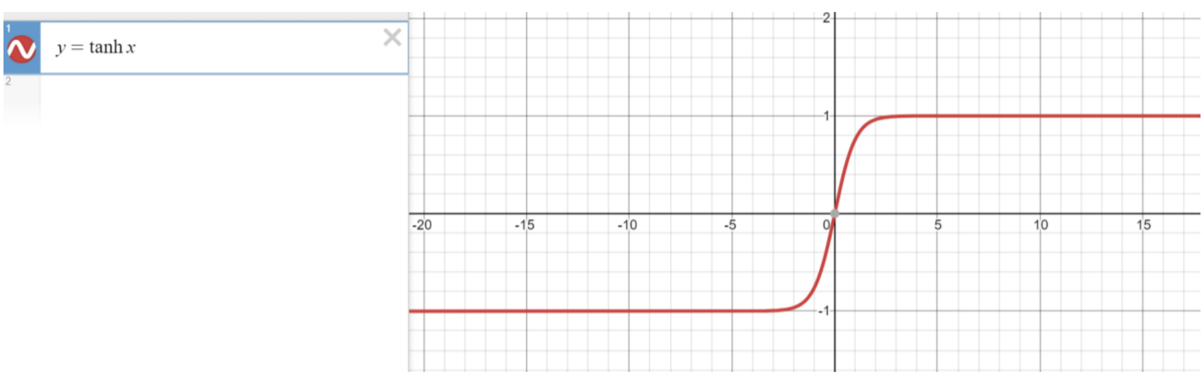
$$\lim_{x\to\infty}\left(2+\frac{1}{x}\right)=2$$

$$\lim_{x\to-\infty}\left(2+\frac{1}{x}\right)=2$$

The limit at infinity is whatever the horizontal asymptote is, if it exists.

Some functions have horizontal asymptotes built in.
Examples: exponential functions (at y=0)
Inverse tangent (inverse hyperbolic tangent is similar), there are two horizontal asymptotes. $\pm\frac{\pi}{2}$ (tanh x goes to ± 1)





Thinking in terms of the squeeze theorem or properties of limits.

$$\lim_{x\to\infty}\frac{-1}{x}\le\lim_{x\to\infty}\frac{\sin x}{x}\le\lim_{x\to\infty}\frac{1}{x}$$

Primarily dealing with horizontal or oblique/slant asymptote case for rational function. Vertical asymptotes are addressed in Chapter 2.

Combine your algebra knowledge with calculus.
When sketching, it is a good idea to start with any horizontal/slant and vertical asymptotes.
But rather doing "test points", use calculus and test points in the derivative to find extreme values (maxima/minima) or concavity.

Sometimes the algebra will be a little messy. Mostly focus on the numerator to find the critical points in the first derivative. And then you may want to test those critical points in the second derivative to find the concavity. (a lot can be done with the first derivative)

We can cover an example with a cusp on Tuesday.