3/15/2022
It's the Ides of March!

4.3 Maxima and Minima (review and finish)
4.4 The Mean Value Theorem (for Derivatives)
4.5 Derivatives and the shape of a graph

Maxima and minima

The first derivative test. We defined critical points as the places where the derivative is either undefined or zero. The first derivative test involved constructing a sign chart. And when a critical is increasing on the left and decreasing on the right that is a maximum. When a critical point is decreasing on the left and increasing on the right then it is a minimum. If the function is not defined at that point, it's not an extremum. If there is no sign change, then it's not an extremum.

The second derivative test:

Concavity: essentially the direction in which the curve of the graph is opening. Does the bowl shape open upward or downward? If the curve is "opening upward" then the graph in that region is concave up. And if the curve is "opening downward" then the graph is concave down.

The point where the concavity changes is called an inflection point. This is where the graph switches from concave up to concave down, or the reverse.

The inflection point corresponds to where the second derivative is equal to zero.

The sign of the second derivative at the critical points (these are from the first derivative) tells you whether the graph is concave up at that point (minimum) or the graph is concave down at the point (corresponds to a maximum).

To test for maxima and minima using the second derivative test:

1) First find the critical points in the first derivative
2) Test the critical point in the second derivative: if the sign is negative the graph is concave down, if the sign is positive, the graph is concave up.

Suppose that our function is $y = x^2$.
The critical is where the first derivative is zero. $y' = 2x$, and $2x = 0$ implies the critical point is at $x = 0$.
The second derivative is $y'' = 2$. The second derivative is always positive. The graph is always concave up. The critical point is a minimum.

Suppose our function $y = x^3$. The first derivative is $y = 3x^2$, and we set that equal to zero: $3x^2 = 0$ implies the critical point is at $x = 0$.
The second derivative is $y'' = 6x$. We plug in the critical point: 6(0)=0. This is undetermined. This is an inflection point. If you think about the graph, this is not an extremum because the sign of the first derivative doesn't change.

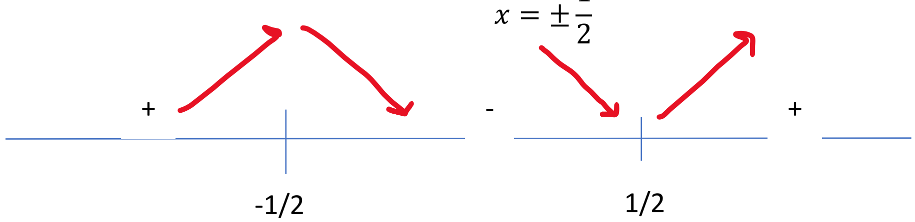
Find the critical points and any extrema for the function $y = 4x^3 - 3x$.

We’re going to look at both the first and second derivative test.

First derivative test:

$$y' = 12x^2 - 3$$
$$12x^2 - 3 = 0$$
$$12x^2 = 3$$
$$x^2 = \frac{1}{4}$$

$$x = \pm\frac{1}{2}$$



Test points are: -1, 0, 1
Test these points in the first derivative to get the signs.

$$y'(-1) = 12(-1)^2 - 3 = 12 - 3 = 9$$
$$y'(0) = 12(0)^2 - 3 = -3$$
$$y'(1) = 12(1)^2 - 3 = 12 - 3 = 9$$

$x = -\frac{1}{2}$ corresponds to a maximum, and $x = \frac{1}{2}$ corresponds to a minimum.

The second derivative test:

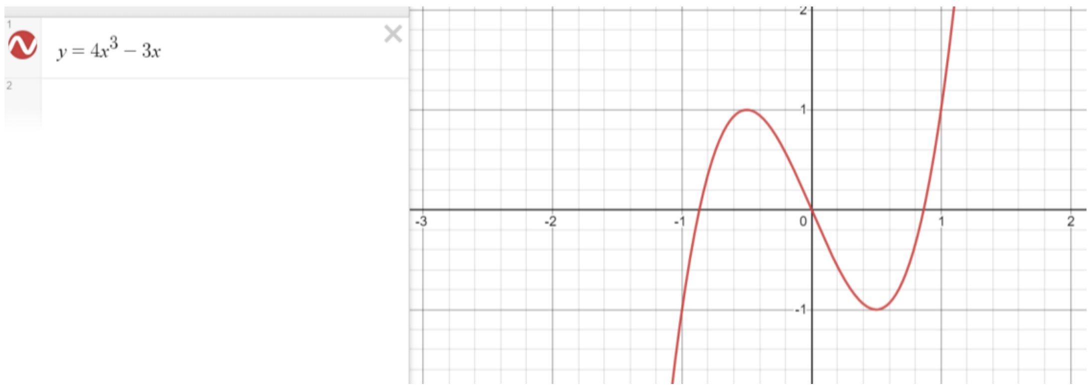
$$y'' = 24x$$

We want to test the critical points in the second derivative.

$$y''\left(-\frac{1}{2}\right) = 24\left(-\frac{1}{2}\right) = -12$$

$$y''\left(\frac{1}{2}\right) = 24\left(\frac{1}{2}\right) = 12$$

The -12 is telling us that the graph at this point is concave down, and so the critical point represents the location of a maximum.
The 12 is telling us that the graph at this point is concave up, and so the critical point represents the location of a minimum.



The function is $f(x) = x^2e^{-x}$.

We want to find the critical points, and then test them for whether they are maxima or minima (classify the critical points).

$$f'(x) = 2xe^{-x} - x^2e^{-x}$$

Factor this to find the critical points.

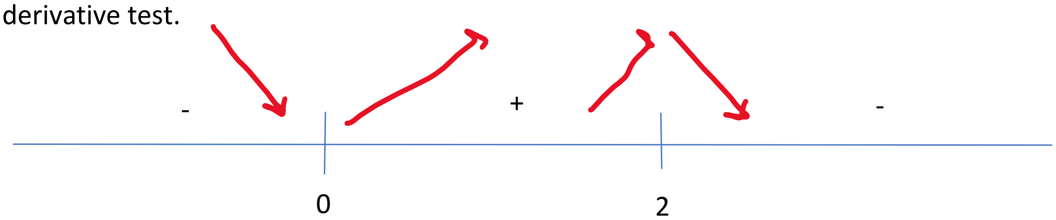
$$f'(x) = e^{-x}(2x - x^2)$$

The e^{-x} factor can never be 0. That means that the critical points have to come out of the other factor.

$$(2x - x^2) = 0$$
$$x(2 - x) = 0$$

Critical points are at $x = 0, x = 2$.

First derivative test.



Test points: -1, 1, 3

$$y'(-1) = (+)(-)(+) = e^{-1}(-1)(3) = (-)$$
$$y'(1) = (+)(+)(+) = e^{1}(1)(1) = (+)$$
$$y'(3) = (+)(+)(-) = e^{3}(3)(-1) = (-)$$

The point $x = 0$ corresponds to a minimum. The point $x = 2$ corresponds to a maximum.

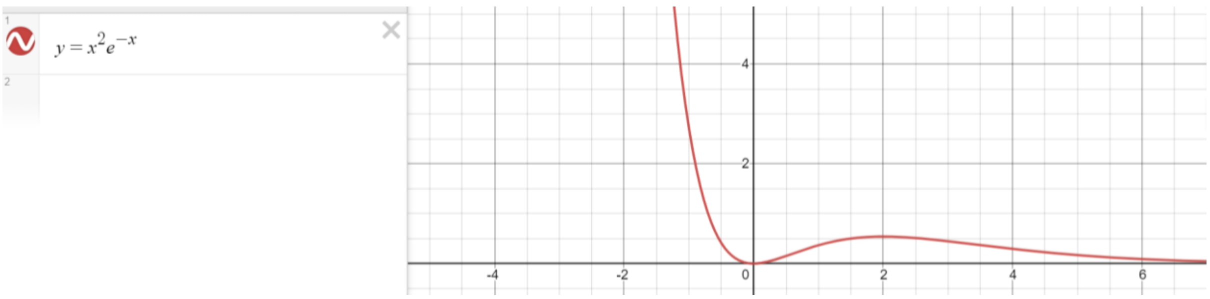
The second derivative test.

$$f''(x) = -e^{-x}(2x - x^2) + e^{-x}(2 - 2x)$$

$$f''(x) = e^{-x}(-2x + x^2 + 2 - 2x) = e^{-x}(x^2 - 4x + 2)$$

This won't factor. But e^{-x} term is still always positive, so the polynomial is going to decide the sign of the whole expression.

Plug in $x = 0$, $f''(0) = e^0(0 - 4(0) + 2) = 2$, so this is concave up = minimum
Plug in $x = 2$, $f''(2) = e^2(2^2 - 4(2) + 2) = e^2(4 - 8 + 2) = -e^2(2)$, so this is concave down = a maximum.



Absolute extreme values.
Absolute maximum: is a point where the function is larger than (or equal to) any other point on the function.
Absolute minimum: is a point where the function is smaller than (or equal to) any other point on the function.

Absolute extrema are not guaranteed to exist in general (specifically on any open interval). But any closed interval is guaranteed to have both an absolute maximum and an absolute minimum.

The problems will include a function, and a closed interval.

$y = x^2 + \frac{2}{x}$ on the interval $[1,4]$.

Note: that there can be no point of discontinuity in the interval for this to work.

Absolute extrema can occur either at the critical points, or at the endpoints of the interval.
We have to find the critical points (using the first derivative), and then keep those critical points that are inside the interval.

Then plug them into the original function and select the largest function value to be the absolute maximum, and the smallest function value to be the absolute minimum.

$$y' = 2x - \frac{2}{x^2}$$

$$2x - \frac{2}{x^2} = 0$$

$$2x = \frac{2}{x^2}$$

$$x^3 = 1$$
$$x = 1$$

This is already one of the endpoints.

Test critical points and endpoints in the function:
$f(1) = 1^2 + \frac{2}{1} = 1 + 2 = 3$, absolute minimum
$f(4) = 4^2 + \frac{2}{4} = 16 + \frac{1}{2} = \frac{33}{2}$, absolute maximum

$y = (x - x^2)^2$ on the interval $[-1,1]$

$$y' = 2(x - x^2)(1 - 2x)$$
$$y' = 2(x)(1 - x)(1 - 2x)$$

Critical points: $x = 0, x = 1, x = \frac{1}{2}$
All these critical points are inside the interval, so check all of them and the endpoints.

$y(-1) = (-1 - (-1)^2)^2 = (-1 - 1)^2 = (-2)^2 = 4$, absolute maximum
$y(0) = 0$, absolute minimum
$y\left(\frac{1}{2}\right) = \left(\frac{1}{2} - \left(\frac{1}{2}\right)^2\right)^2 = \left(\frac{1}{2} - \frac{1}{4}\right)^2 = \left(\frac{1}{4}\right)^2 = \frac{1}{16}$ (nothing)
$y(1) = (1 - 1^2)^2 = 0$, absolute minimum

The main thing to not forget is to discard critical points that are not inside the interval.

Mean Value Theorem (for derivatives)

Before we get there:

Rolle's Theorem: if f is continuous on a closed interval $[a, b]$, and differentiable on the corresponding open interval, and $f(a) = f(b)$ then there exists at least one point c inside the interval (a, b) such that $f'(c) = 0$.

This is theorem is required to prove the Mean Value Theorem (which we won't do here).

Mean Value Theorem: If f is a function continuous on $[a, b]$ and differentiable on the corresponding open interval, then there exists at least one point c in the open interval where
$f'(c)$ is equal to the slope between $\left(a, f(a)\right)$ and $\left(b, f(b)\right)$.

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Example.
$f(x) = \sqrt{x}$, the interval is [0,9].

Is it continuous on the interval? Yes.
Is it differentiable (continuous derivative on the open interval)? Yes (since 0 is not in the open interval)

$$y' = \frac{1}{2\sqrt{x}}$$

What slope does it have to be equal to?

$f(0) = 0$, corresponding point is $(0,0)$
$f(9) = \sqrt{9} = 3$, corresponding point is $(9,3)$

What is the slope between these two points?

$$m = \frac{3-0}{9-0} = \frac{1}{3}$$

We want to find the point where the derivative is equal to 1/3.

$$\frac{1}{2\sqrt{x}} = \frac{1}{3}$$

$$3 = 2\sqrt{x}$$
$$\frac{3}{2} = \sqrt{x}$$

$$\frac{9}{4} = x$$

The point $x = \frac{9}{4}$ is inside the interval $[0,9]$ and is the point where the slope of the derivative is equal to the average rate of change between the endpoints (the slope of the secant line).

Use derivatives and properties of derivatives to sketch graphs of functions.
We talked about the first derivative test: use that to determine critical points, and where the function is increasing or decreasing based on the sign chart.
We talked about concavity, and how the graph is curving based on the sign of the second derivative.
We can use this information (along with some old algebra) to better graph a function without a calculator. (asymptotes+intercepts)