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3.4 Derivatives as Rates of Change
3.5 Derivatives of Trig Functions

Estimating the value of a function using the derivative… linear estimation. Only applies "near" a known value. The further away you get, the less good the estimation will be.

Estimating along the tangent line. The derivative gives us the slope, and the "nice" point on the curve that we know is our starting point. And then we move along the tangent a certain distance to obtain the estimate of the change.

The change in y is approximately the slope (value of the derivative) times the amount of change in x.

$$\Delta y \approx f'(a)\Delta x$$

$$y + \Delta y = f(a) + f'(a)\Delta x$$

A function $f(x) = x^2$. Use derivatives to estimate the value of $f(3.2)$.

$$a = 3, \Delta x = 0.2$$

$$f(a): f(3) = 9 = y$$
$$\Delta y = f'(3)\Delta x = (6)(0.2) = 1.2$$

$$f'(x) = 2x, f'(3) = 6$$
$$f(3.2) \approx y + \Delta y = f(a) + f'(a)\Delta x = 9 + (6)(0.2) = 9 + 1.2 = 10.2$$

Close to 10.24.

When we talk about "differentials" this kind of example will come back again.

Another way of asking this question is estimate with derivatives (differentials) the values of 3.2^2.

We are estimating near a nice value (here it's an integer). The nice value is 3. The change in x, the differential part, Δx is the difference between the nice value and the one given. Infer the function from the operation being done to the number… here that's $f(x) = x^2$.

If the problem had asked about estimating $\sqrt{3.8}$… the function is the square root function. And the nice value is 4. And the Δx is -0.2.

Position-Velocity-Acceleration
Position function $s(t)$. The velocity of a particle traveling along $s(t)$ is $v(t) = s'(t)$ as function. Remember that the average velocity is the slope of the secant line (the slope between the two specified points). Average rate of change. If they want the instantaneous velocity, then they will only give you one point. The value of the derivative at that point.

Acceleration of a particle traveling along $s(t)$, then $a(t) = v'(t) = s''(t)$. Acceleration is the rate of change of the velocity. Average acceleration (needs two points on the velocity function), and the instantaneous acceleration (the value of the second derivative at the point provided).

Population change (linear approximation).
The population is currently 10,000. Triples every five years. What is estimated population in 2 years.

At time =0, P=10 (in thousands)
At time = 5, P=30 (in thousands)
Slope between these values $\frac{30-10}{5-0} = \frac{20}{5} = 4$

Over a 5-year period (within this first five-year period), the average rate of change is about 4 thousand per year.

Estimate the increase 4(2) (it's over two years)… 8. 8 more than the previous 10 is 18 thousand.

(this is a terrible estimate – population doesn't grow linearly).

Marginal Revenue/Cost/Profit.

Marginal Cost = $MC(x) = C'(x)$ where $C(x)$ is the cost.
Marginal Revenue = $MR(x) = R'(x)$ where $R(x)$ is the revenue function
Marginal Profit = $MP(x) = P'(x)$ where $P(x) = R(x) - C(x)$ is the profit function.

Suppose we have a price function $p(x) = 9 - 0.03x$.
The revenue function $R(x) = xp(x) = 9x - 0.03x^2$
What is the marginal Revenue when $x = 100$?
$R'(x) = 9 - 0.06x$, $R'(100) = 9 - 0.06(100) = 9 - 6 = 3$

Marginal Revenue is the change in revenue when x number of items are sold. If the company sells 100 items, the next item will bring in \$3 of **additional** revenue. (additional = change in)

Trig Function Derivatives

Finding the derivative of sin(x).

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\sin\,(x+h) - \sin\,(x)}{h}$$

$$\sin(a+b) = \sin a \cos b + \cos a \sin b$$
$$\cos(a+b) = \cos a \cos b - \sin a \sin b$$

$$= \lim_{h\to 0} \frac{\sin x \cos h + \cos x \sin h - \sin(x)}{h} = \lim_{h\to 0} \frac{\sin x \cos h - \sin(x)}{h} + \frac{\cos x \sin h}{h} =$$

$$\lim_{h\to 0} \left[\sin\,(x)\left(\frac{\cos h - 1}{h}\right) + \cos\,(x)\left(\frac{\sin h}{h}\right)\right] = (\sin x) \lim_{h\to 0} \frac{\cos h - 1}{h} + (\cos x) \lim_{h\to 0} \frac{\sin h}{h}$$

$$\lim_{h\to 0}\frac{\cos h - 1}{h} = 0, \lim_{h\to 0}\frac{\sin h}{h} = 1$$

$$= (\sin x)(0) + (\cos x)(1) = \cos x$$

$$\frac{d}{dx}[\sin x] = \cos x$$

$$\frac{d}{dx}[\cos x] = -\sin x$$

The derivative of the tangent function. $\tan(x) = \frac{\sin(x)}{\cos(x)}$, using the quotient rule.

$$h'(x) = \frac{f'(x)g(x) - g'(x)f(x)}{[g(x)]^2}$$

$f(x) = \sin(x)$	$g(x) = \cos x$
$f'(x) = \cos(x)$	$g'(x) = -\sin(x)$

$$\frac{d}{dx}[\tan x] = \frac{\cos x \cos x - (-\sin x)\sin x}{\cos^2(x)} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$$

$$\cot(x) = \frac{\cos(x)}{\sin(x)}$$

$$\frac{d}{dx}[\cot x] = -\csc^2 x$$

$$\frac{d}{dx}[\sec x] = \frac{d}{dx}\left[\frac{1}{\cos x}\right] =$$

$f(x) = 1$	$g(x) = \cos x$
$f'(x) = 0$	$g'(x) = -\sin x$

$$\frac{0(\cos x) - (-\sin x)(1)}{\cos^2 x} = \frac{\sin(x)}{\cos^2 x} = \frac{\sin(x)}{\cos(x)} \cdot \frac{1}{\cos x} = \sec x \tan x$$

$$\frac{d}{dx}[\csc x] = \frac{d}{dx}\left[\frac{1}{\sin x}\right] = -\csc x \cot x$$

Higher-order derivatives (for sine and cosine).

$$\frac{d}{dx}[\sin x] = \cos x$$
$$\frac{d}{dx}[\cos x] = -\sin x$$
$$\frac{d}{dx}[-\sin x] = -\cos x$$
$$\frac{d}{dx}[-\cos x] = \sin x$$

There is a cycle of 4 derivatives, and these functions will repeat.

$$\frac{d^{25}}{dx^{25}}[\sin x] = \frac{d}{dx}\left[\frac{d^{24}}{dx^{24}}[\sin x]\right] = \frac{d}{dx}[\sin x] = \cos x$$

Chain Rule

Is a rule we use for composed functions $f\big(g(x)\big)$. Examples are $(3x+1)^3$, $\sin(x^3)$, $\sqrt{x^2+1}$, $\sec^2 x$, $etc.$

$f\big(g(x)\big) = (f \circ g)(x) = f(u)$ where $u = g(x)$

$$h(x) = f\big(g(x)\big) = f(u)$$
$$h'(x) = f'(u)\frac{d}{dx}(u) = \frac{df}{du}\cdot\frac{du}{dx} = f'(g(x))\cdot g'(x)$$

$$h(x) = (3x+1)^5$$

$$f(x) = u^5, g(x) = 3x+1$$

$$h'(x) = 5(u)^4 g'(x) = 5(3x+1)^4(3) = 15(3x+1)^4$$

Test with a square function.

$$h(x) = \sec^2 x$$

product rule: $h(x) = \sec x \sec x$

$f(x) = \sec x$	$g(x) = \sec x$
$f'(x) = \sec x \tan x$	$g'(x) = \sec x \tan x$

$$h'(x) = \sec x\,(\sec x \tan x) + (\sec x \tan x)(\sec x) = 2\sec^2 x \tan x$$

Using the chain rule:

$$h(x) = \sec^2 x$$
$$f(u) = u^2, g(x) = \sec x$$

$$f'(u)g'(x) = 2(u)^1 \sec x \tan x = 2\sec x \sec x \tan x = 2\sec^2 x \tan x$$

$$h(x) = \sqrt{x^2 + 1}$$

$$f(u) = \sqrt{u} = u^{1/2}, g(x) = x^2 + 1$$

$$f'(u) = \frac{1}{2}u^{-\frac{1}{2}}(2x) = \frac{x}{\sqrt{x^2 + 1}}$$

$$h(x) = \sin(x^3)$$

$$f(u) = \sin(u)\,, g(x) = x^3$$

$$f'(u)g'(x) = \cos(u)\,(3x^2) = 3x^2\cos\,(x^3)$$

A common mistake: say $h'(x) = \cos\,(3x^2)$… THIS IS WRONG.
Don't the take the derivative inside the function. It multiplies on the outside.

One three-level example.

$$F(x) = f\left(g\left(h\big(j(x)\big)\right)\right) = \csc(1 + \cos^2 x^3)$$

$$f(x) = \csc(u)$$
$$u = g(x) = 1 + \cos^2 x^3 = 1 + v^2$$
$$v = h(x) = \cos(x^3) = \cos\,(w)$$
$$w = j(x) = x^3$$

$$F'(x) = f'(u)g'(v)h'(w)j'(x)$$

$$F'(x) = -\csc u \cot u \cdot (2v) \cdot (-\sin(w)) \cdot 3x^2 =$$

$$-\csc(1 + \cos^2 x^3)\cot(1 + \cos^2 x^3) \cdot (2\cos(x^3)) \cdot (-\sin(x^3)) \cdot 3x^2$$

Next time, we'll also look how to derive the quotient rule from the chain rule.

Continue with more chain rule examples.