1/25/2022

2.3 Limit Laws, Evaluating Limits using Algebra

Example of applying limit laws:

$$\lim_{x\to 1}\frac{x^2-3x+7}{x^2+3}=\frac{\lim_{x\to 1}x^2-3x+7}{\lim_{x\to 1}x^2+3}=\frac{1-3+7}{1+3}=\frac{5}{4}$$

Other algebraic techniques we can use to simplify expressions to find the limit.

$$\lim_{x\to 2}\frac{x^2-4}{x-2}=\lim_{x\to 2}\frac{(x-2)(x+2)}{x-2}=\lim_{x\to 2}x+2=4$$

Rationalize the expression (involving square roots).

$$\lim_{x\to 1}\frac{x-1}{\sqrt{x}-1}=\lim_{x\to 1}\frac{x-1}{\sqrt{x}-1}\cdot\frac{\sqrt{x}+1}{\sqrt{x}+1}=\lim_{x\to 1}\frac{(x-1)\left(\sqrt{x}+1\right)}{x+\sqrt{x}-\sqrt{x}-1}=\lim_{x\to 1}\frac{(x-1)\left(\sqrt{x}+1\right)}{x-1}=$$

$$\lim_{x\to 1}\left(\sqrt{x}+1\right)=2$$

Example

$$\lim_{x\to -1}\frac{\sqrt{x+2}-1}{x+1}=\lim_{x\to -1}\frac{\sqrt{x+2}-1}{x+1}\cdot\frac{\sqrt{x+2}+1}{\sqrt{x+2}+1}=\lim_{x\to -1}\frac{\left(\sqrt{x+2}\right)^2-1^2}{(x+1)\left(\sqrt{x+2}+1\right)}=$$

$$\lim_{x\to -1}\frac{x+2-1}{(x+1)\left(\sqrt{x+2}+1\right)}=\lim_{x\to -1}\frac{x+1}{(x+1)\left(\sqrt{x+2}+1\right)}=\lim_{x\to -1}\frac{1}{\left(\sqrt{x+2}+1\right)}=\frac{1}{2}$$

Simplifying complex fractions

$$\lim_{x\to 1}\frac{\frac{1}{x+1}-\frac{1}{2}}{x-1}=\lim_{x\to 1}\frac{\frac{1}{(x+1)}\cdot\frac{2}{2}-\frac{1}{2}\cdot\frac{x+1}{x+1}}{x-1}=\lim_{x\to 1}\frac{\frac{2}{2(x+1)}-\frac{x+1}{2(x+1)}}{x-1}=$$

$$\lim_{x\to 1}\frac{\frac{2-(x+1)}{2(x+1)}}{x-1}=\lim_{x\to 1}\frac{\frac{-x+1}{2(x+1)}}{x-1}=\lim_{x\to 1}\frac{\frac{-(x-1)}{2(x+1)}}{x-1}=\lim_{x\to 1}\frac{-(x-1)}{2(x+1)}\div(x-1)=$$

$$\lim_{x\to 1}\frac{-(x-1)}{2(x+1)}\cdot\frac{1}{x-1}=\lim_{x\to 1}\frac{-1}{2(x+1)}=-\frac{1}{4}$$

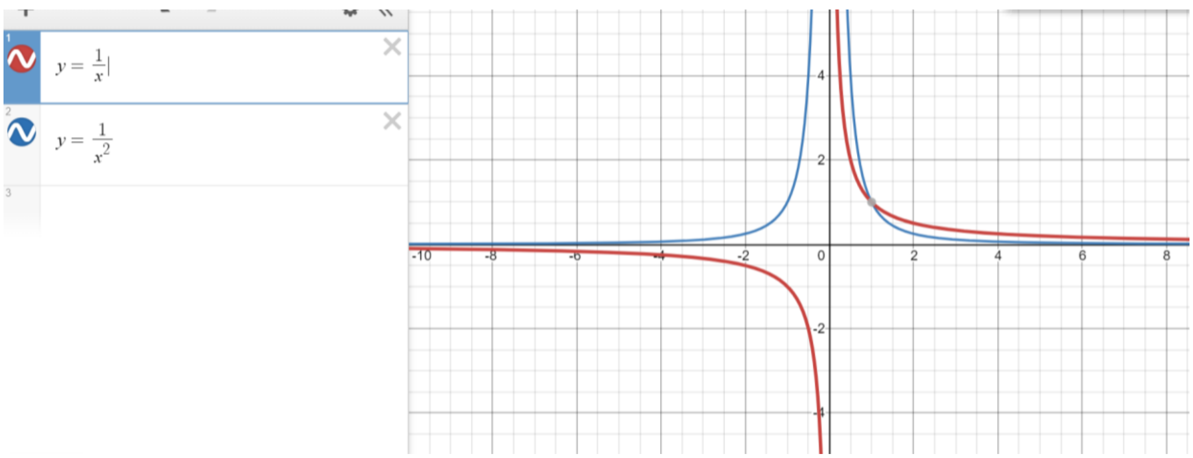
Complex set of expressions with addition or subtraction between them, and one or both are undefined (limit laws don't apply). In this example, we are going to combine the terms, each of which is behaving badly, but together may behave better.

$$\lim_{x\to 0}\left(\frac{1}{x}+\frac{5}{x(x-5)}\right)=\lim_{x\to 0}\left(\frac{1}{x}\cdot\frac{x-5}{x-5}+\frac{5}{x(x-5)}\right)=\lim_{x\to 0}\left(\frac{x-5+5}{x(x-5)}\right)=$$

$$\lim_{x\to 0}\left(\frac{x}{x(x-5)}\right)=\lim_{x\to 0}\left(\frac{1}{(x-5)}\right)=-\frac{1}{5}$$

Example with infinity

$$\lim_{x\to 1}\frac{x+2}{(x-1)^2}=\frac{+}{0}=+\infty$$



Infinite case determined by even powers in the denominator, and the sign of the numerator. If the denominator is an odd power, the sign changes as you go through zero, and so the result will be DNE (does not exist) because the infinities will have different signs.

Squeeze Theorem

$$\lim_{x\to 0}x(-1)\le\lim_{x\to 0}x\sin\left(\frac{1}{x}\right)\le\lim_{x\to 0}x(1)$$

This is true because $-1\le\sin\left(\frac{1}{x}\right)\le 1$

What I know is that $\lim_{x\to 0}x(-1)=0$, and that $\lim_{x\to 0}x(1)=0$

$$0\le\lim_{x\to 0}x\sin\left(\frac{1}{x}\right)\le 0$$

$$\lim_{x\to 0}x\sin\left(\frac{1}{x}\right)=0$$

Typically, this is applied when the limit goes to zero.

If the limit is a value other than zero, to apply the squeeze theorem you could "squeeze" the expression $\lim_{x \to c} f(x) - L$.

Special limit:

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

$$\lim_{x \to 0} \frac{1 - \cos x}{x} = 0$$

2.4 Continuity
What is a continuous function?
If you can draw the function without picking up your pencil, then the function is continuous.

Where are functions not continuous? Holes. Vertical Asymptotes. Jump discontinuities.

Formal definition of continuity – continuity at a point
1. The limit has to exist at the point. (for the limit to exist, the right and left-handed limits have to agree)
2. The function has to be defined at that same point.
3. The function value and the limit value have to agree.

When you test for continuity, you have to test for these three things.

Types of discontinuities:
1) Removable discontinuity: at a hole
2) Infinite discontinuity: vertical asymptote
3) Jump discontinuity: the difference between the two pieces is a finite value, but can't be filled in with a single point (the one-sided limits are finite and don't agree)

The easiest way to classify discontinuities is to graph the function.

From the definition of continuity at a point, we can extend this to continuity on an interval.
A function is continuous if it is continuous at every point in its domain, and the domain has no holes (rational function). Some functions are considered continuous if they are "one-sided continuous" on one endpoint of the domain.

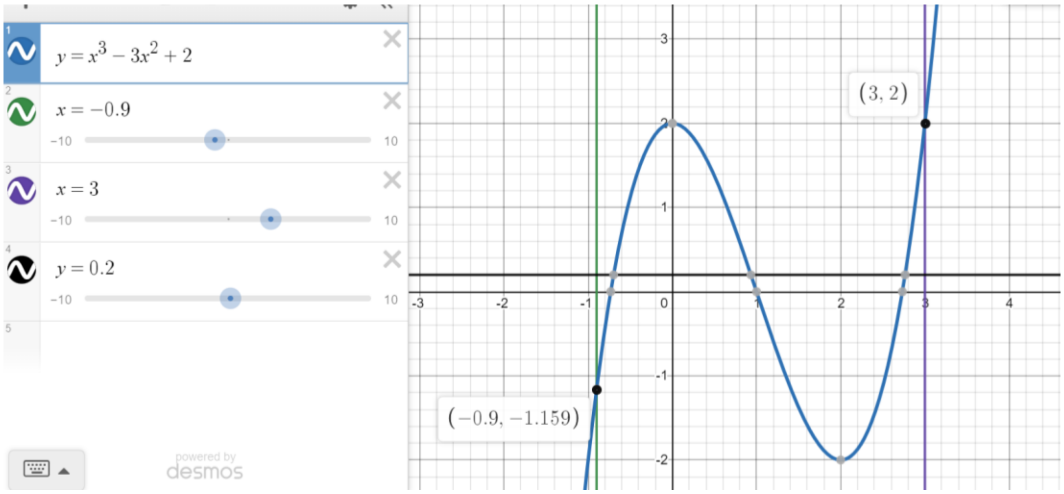
$$f(x) = \sqrt{x}$$

At x=0, there is no two-sided limit. There is only the right-sided limit at 0.

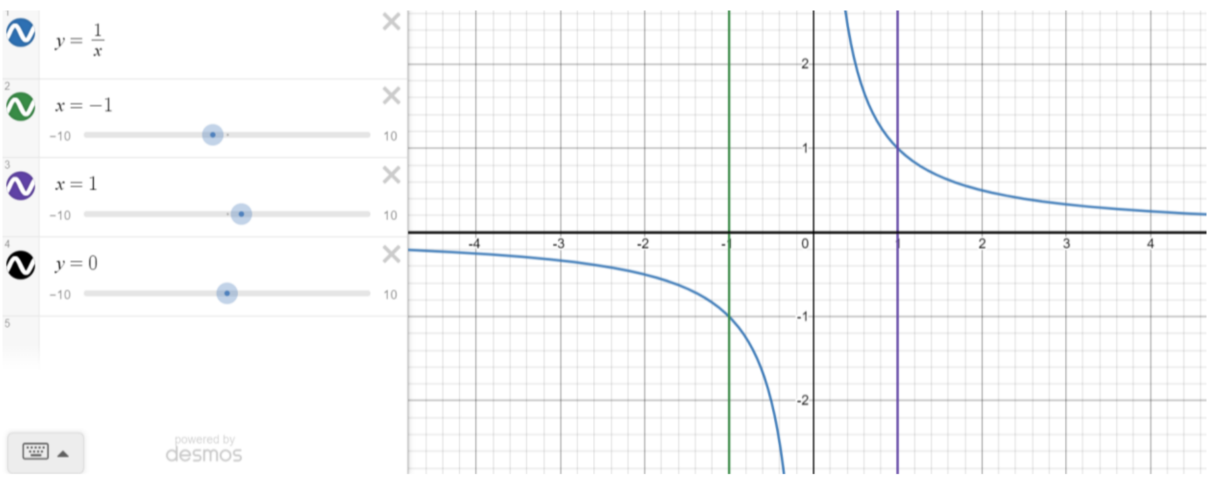
For $f(x) = \sqrt{4 - x^2}$, the domain is [-2,2].
The one-sided continuity applies at both ends of the domain.

Intermediate Value theorem

If a function is continuous on a closed interval [a,b], and there is a value z between f(a) and f(b), then there is some number c in the interval [a,b], such that f(c)=z.



Here the intermediate value theorem applies because the function is continuous.



This example the function is not continuous, and so the theorem fails to apply.