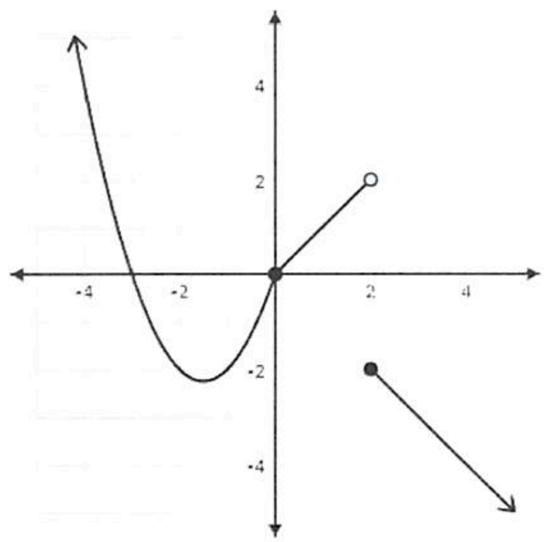
MTH 263, Exam #1, Spring 2022 Name KEY

Instructions: Work the problems below as directed. Show all work. Clearly mark your final answers. Use exact values unless the problem specifically directs you to round. Simplify as much as possible. Partial credit is possible, but solutions without work will not receive full credit.

Part 1: These questions you will submit answers to in Canvas. Show all work and submit the work with Part 2 of the exam. But you must submit the answers in Canvas to receive credit. Each question/answer will be listed separately. The Canvas question will refer to the number/part to indicate where you should submit which answer. The questions will appear in order (in case there is an inadvertent typo). Correct answers will receive full credit with or without work in this section, but if you don't submit work and clearly label your answers, you won't be able to challenge any scoring decisions for making an error or any kind.

1. Find the indicated limits.
   a. $\lim_{x\to 3} \frac{x^3-27}{x-3}$ (8 points) $\lim_{x\to 3} \frac{\cancel{(x-3)}(x^2+3x+9)}{\cancel{x-3}} = \lim_{x\to 3} x^2+3x+9 = 9+9+9 = \boxed{27}$

   b. The following questions refer to the graph of $f(x)$ shown. (5 points each)



   i. $\lim_{x\to 0^+} f(x) = 0$

   ii. $\lim_{x\to 0^-} f(x) = 0$

   iii. $\lim_{x\to 0} f(x) = 0$

   iv. $\lim_{x\to 2^-} f(x) = 2$

   v. $\lim_{x\to 2^+} f(x) = -2$

   vi. $\lim_{x\to 2} f(x) =$ DNE

   vii. Is the graph continuous at $x = 0$? yes

   viii. Is the graph continuous at $x = 2$? no

2. A particle is moving along a trajectory defined by $s(t) = t^3 - 3t + 7$. Find the average velocity on the interval $[0,2]$. (10 points)

$s(0) = 0 - 0 + 7 = 7$
$s(2) = 8 - 6 + 7 = 9$

$\frac{9-7}{2-0} = \frac{2}{2} = \boxed{1}$

3. Find the value of the indicated derivative. (8 points each)

a. $f(x) = 3x^3 - \frac{4}{x^2}, f'(1)$ $\quad 3x^3 - 4x^{-2}$

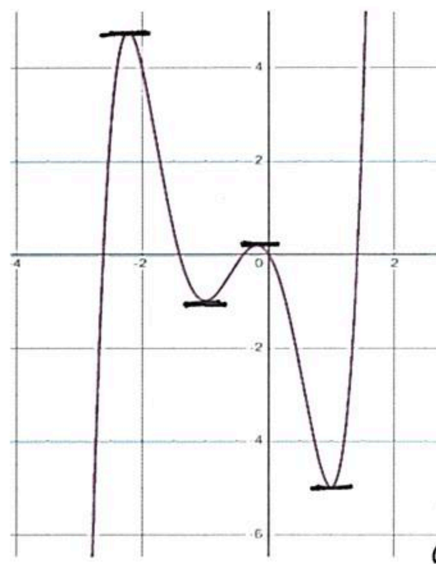
$f'(x) = 9x^2 + 8x^{-3} = 9x^2 + \frac{8}{x^3} \qquad f'(1) = 9(1)^2 + \frac{8}{1^3} = 9 - 8 = \boxed{17}$

b. $f(x) = (4 - x^2)^4, f''(-1)$

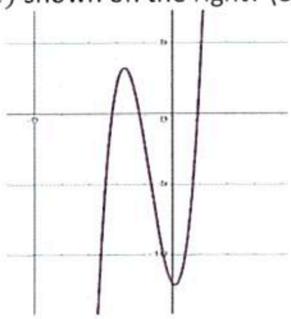
$f'(x) = 4(4-x^2)^3(-2x) = -8x(4-x^2)^3 \qquad f''(-1) = -8(3)^3 + 48(1)(3)^2 = \boxed{216}$

$f''(x) = -8(4-x^2)^3 - 8x \cdot 3(4-x^2)^2(-2x) = -8(4-x^2)^3 + 48x^2(4-x^2)^2$

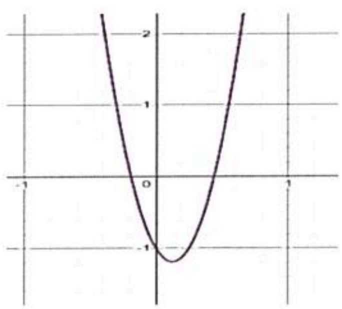
4. Which of the following graphs is most likely to be the graph of the derivative of the graph of $f(x)$ shown on the right? (8 points)



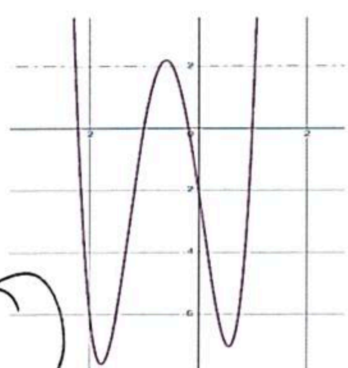
4 zeros in deriv.



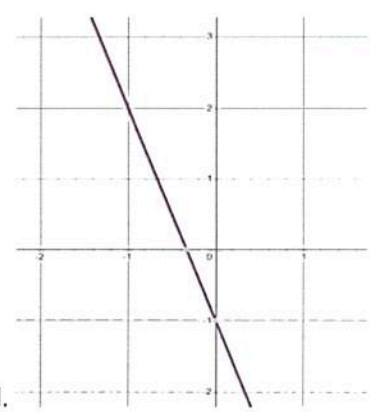
a.



b.



c.



d.

Part 2: In this section you will record your answers on paper along with your work. After scanning, submit them to a Canvas dropbox as directed. These questions will be graded by hand.

5. Find the indicated limits.

a. $\lim_{x \to 16} \frac{x-16}{\sqrt{x}-4}$ (12 points) $\lim_{x \to 16} \frac{(x-16)(\sqrt{x}+4)}{(\sqrt{x}-4)(\sqrt{x}+4)} = \lim_{x \to 16} \frac{\cancel{(x-16)}(\sqrt{x}+4)}{\cancel{x-16}} = \lim \sqrt{x}+4 =$

$4+4 = \boxed{8}$

b. $\lim_{x \to 0} \frac{\cos(2x)-\cos(5x)}{x^2}$ [Hint: Find this one numerically. Show your test points.] (15 points)

See attached Excel file

$= 10.5$

$= 21/2$

The general rule here

is $\lim_{x \to 0} \frac{\cos(ax) - \cos(bx)}{x^2} = \frac{b^2 - a^2}{2}$

6. Use the squeeze theorem to prove that $\lim_{x \to 0} x^3 \sin\left(\frac{1}{x}\right) = 0$. (15 points)

$-1 \leq \sin\left(\frac{1}{x}\right) \leq 1$

$-1(x^3) \leq x^3 \sin\left(\frac{1}{x}\right) \leq x^3(1)$

$\lim_{x \to 0} -x^3 \leq \lim_{x \to 0} x^3 \sin\left(\frac{1}{x}\right) \leq \lim_{x \to 0} x^3$

$0 \leq \lim_{x \to 0} x^3 \sin\left(\frac{1}{x}\right) \leq 0$

$\therefore \lim_{x \to 0} x^3 \sin\left(\frac{1}{x}\right) = 0$

7. A particle is moving along a trajectory defined by $s(t) = t^2 - 3t + 7$. Find the instantaneous velocity at the point $t = 1$ using the limit definition of the derivative. You should provide both the general equation, and the velocity at that point. (15 points)

$$s'(t) = \lim_{h\to 0} \frac{(t+h)^2 - 3(t+h) + 7 - (t^2-3t+7)}{h} =$$

$$\lim_{h\to 0} \frac{\cancel{t^2} + 2ht + h^2 - \cancel{3t} - 3h + \cancel{7} - \cancel{t^2} + \cancel{3t} - \cancel{7}}{h} =$$

$$\lim_{h\to 0} \frac{2ht + h^2 - 3h}{h} = \lim_{h\to 0} \frac{\cancel{h}(2t+h-3)}{\cancel{h}} = \lim_{h\to 0} 2t + h - 3 =$$

$$= \boxed{2t-3}$$

$$s'(1) = 2(1) - 3 = \boxed{-1}$$

8. Use the $\varepsilon - \delta$ definition of the limit to prove that $\lim_{x\to 1}(2x+7) = 9$. (15 points)

$|x-1| < \delta \qquad |2x+7-9| < \varepsilon \to |2x-2| < \varepsilon \to 2|x-1| < \varepsilon \to |x-1| < \frac{\varepsilon}{2} = \delta$

if $|x-1| < \delta$, then let $\delta = \varepsilon/2$, we have $|x-1| < \varepsilon/2$ or $2|x-1| < \varepsilon$. we rewrite $|2x-2| < \varepsilon$ and rewriting again we have $|(2x+7)-9| < \varepsilon$. Therefore $\lim_{x\to 1} 2x+7 = 9$

9. Find the derivative of each of the following. If no derivative notation is specified, find the first derivative only. If a derivative notation is specified with the function, find the indicated derivative. (10 points each)

a. $f(x) = e^{\cos x}$

$$f'(x) = e^{\cos x}(-\sin x) = -\sin x \cdot e^{\cos x}$$

b. $f(x) = x \tan x, f''(x)$

$$f'(x) = \tan x + x\sec^2 x$$

$$f''(x) = \sec^2 x + \sec^2 x + x \cdot 2\sec x \sec x \tan x$$

$$= 2\sec^2 x + 2x\sec^2 x \tan x$$

c. $f(x) = \ln(arctan\, x)$

$$f'(x) = \frac{1}{\arctan x} \cdot \frac{1}{1+x^2}$$

d. $xy^2 + y = xy\sin x, \frac{dy}{dx}$

$$y^2 + x \cdot 2yy' + y' = xy\cos x + y\sin x + x\sin x\, y'$$

$$2xyy' + y' - x\sin x\, y' = xy\cos x + y\sin x - y^2$$

$$y'(2xy + 1 - x\sin x) = xy\cos x + y\sin x - y^2$$

$$\boxed{\frac{dy}{dx} = \frac{xy\cos x + y\sin x - y^2}{2xy + 1 - x\sin x}}$$

e. $f(x) = (\tan x)^x$ $\quad y = (\tan x)^x \rightarrow \ln y = x \ln \tan x$

$$\frac{1}{y} y' = \ln(\tan x) + x \cdot \frac{1}{\tan x} \cdot \sec^2 x$$

$\cancel{\cot x}\ \sec^2 x =$

$$\frac{\cos x}{\sin x} \cdot \frac{1}{\cos^2 x} = \frac{1}{\sin x \cos x}$$

$= \csc x \sec x$

$$y' = \frac{dy}{dy} = y(\ln(\tan x) + x\sec x\csc x) =$$

$$\boxed{(\tan x)^x[\ln(\tan x) + x\sec x\csc x]}$$

f. $f(x) = 3^x - \log_5(x^2 + e)$

$$f'(x) = 3^x \ln 3 - \frac{1}{\ln 5} \cdot \frac{1}{x^2+e} \cdot 2x$$

$$= \boxed{(\ln 3)3^x - \frac{2x}{(\ln 5)(x^2+e)}}$$

10. Find the equation of the tangent line to graph $y = \sqrt[3]{x^2 + 4}$ at $x = 2$. (15 points)

$$y' = \frac{1}{3}(x^2+4)^{-2/3} \cdot 2x \qquad (x^2+4)^{1/3}$$

$$y'(2) = \frac{1}{3} \frac{(2)(2)}{\sqrt[3]{(2^2+4)^2}} = \frac{1}{3} \frac{4}{\sqrt[3]{(8)^2}} = \frac{1}{3} \frac{4}{2^2} = \frac{1}{3} \cdot \frac{4}{4} = \frac{1}{3}$$

$$y(2) = \sqrt[3]{2^2+4} = \sqrt[3]{8} = 2$$

$m = 1/3$, pt $(2,2)$

$$\boxed{y - 2 = \frac{1}{3}(x-2)} \rightarrow y - 2 = \frac{1}{3}x - \frac{2}{3} \rightarrow y = \frac{1}{3}x + \frac{4}{3}$$

$+2 \qquad +2$

11. Based on the graph of $f(x)$ shown, sketch a graph of its derivative $f'(x)$. (15 points)