4/5/2022

Chapter 4: Discrete Random Variables
Probability distribution (pdf) for a discrete random variable
Expected value and standard deviation (include weighted averages)
Binomial Distribution
Briefly: other discrete distributions
Counting Methods
Exam #1 is this weekend

Counting Methods:

1. Multiplication Rule
2. Permutations
3. Combinations

Multiplication Rule: If you have a combination of independent events, the total number of elements in the sample space is the product of the number of elements in each independent event.

I toss 3 coins. How many outcomes are there? 8
2*2*2 = 8

How many outcomes are there for flipping two coins with rolling two standard dice?
2*2*6*6=144 outcomes

How many different license plates of the form AAA111 are possible?
26*26*26*10*10*10 assuming all letters and numbers can be used.

What is the probability of getting a license with all even numbers?
(26*26*26*5*5*5)/(26*26*26*10*10*10)

Permutations
When you can't reuse an element more than once. But, the order of the selection matters.
Picking players for positions on a team
Picking lottery winners with different prizes
Picking people for positions with rank

Suppose you have a 14-person t-ball team and the coach is selecting player for the field, and does so randomly (he needs 10 players).
14*13*12*11*10*9*8*7*6*5 = P(14,10), in calculator:
P(n,r)= nPr. So in calculator: 14 nPr 10
Roughly 3.6 billion possibilities

Factorial: 7! = 7*6*5*4*3*2*1 = 5040
P(n,r)=$\frac{n!}{(n-r)!}$

$$P(14,10) = \frac{14!}{(14-10)!} = \frac{14!}{4!} = \frac{14 \times 13 \times 12 \times 11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{4 \times 3 \times 2 \times 1}$$

$$14 \times 13 \times 12 \times 11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5$$

Combinations:
Picking things without reusing them, but you don't care about the order
Picking people for a committee
Picking for a lottery where the prize is the same
"Couples"
Card hands

C(n,r)
nCr (in calculator)

$${}_{r}^{n}C, C_r^n$$
$$\binom{n}{r}$$

Suppose I have 30 employees and as a Christmas bonus, the boss is giving out 5 vacations to Las Vegas in a lottery. How many different ways can those tickets be given out?

$$C(30,5) = \binom{30}{5}$$

$$C(n,r) = \frac{n!}{(n-r)!\,r!}$$

$$\frac{30!}{(30-5)!\,5!} = \frac{30!}{25!\,5!} = \frac{30 \times 29 \times 28 \times \ldots \times 5 \times 4 \times 3 \times 2 \times 1}{25 \times 24 \times 23 \times \ldots . \times 3 \times 2 \times 1 \times 5 \times 4 \times 3 \times 2 \times 1}$$

$$= \frac{30 \times 29 \times 28 \times 27 \times 26}{5 \times 4 \times 3 \times 2 \times 1} = \frac{P(30,5)}{5!} = 6 \times 29 \times 7 \times 9 \times 13 =$$

Combinations are always smaller than permutations.
About 142,000 combinations here (exact number in Excel)

In 5-card poker, a three-of-a-kind is when you have exactly three cards that are the same type (not suit).
4 suits (hearts, diamonds, spades, clubs)
13 card types (ace, 2, 3,4,5,6,7,8,9,10, jack, queen, king)
What is the probability of getting a three-of-a-kind in kings?
How many total hands are possible? How many combinations of 5 cards are there?
$\binom{52}{5} = C(52,5) =$ 2,598,960

Then we want to know how many ways can we get three kings? How many hands can contain three kings?

$$\binom{4}{3}\binom{48}{2} = 4 \times 1128 = 4512$$

Divide to get the probability

$$\frac{\binom{4}{3}\binom{48}{2}}{\binom{52}{5}} = \frac{4512}{2598960} = 0.174\%$$

What about any three-of-a-kind?

$$13 \times \frac{\binom{4}{3}\binom{48}{2}}{\binom{52}{5}} = 2.26\%$$

There are other rules for other situations.

Probability distributions for discrete random variables.
X (capital) is used for "random variable", and x (lower case) is for referring to the value of the variable or the outcome of an experiment

Probability distribution functions (pdf) can sometimes be written as equations, but in the discrete case, they can also just be expressed as tables.

x	0	1	2	3	4	5	6	7	8
P(x)	0.1	0.1	0.1	0.15	0.15	0.1	0.1	0.1	0.1

Cumulative distribution function (cdf) gives the probability of getting the given value or less.

x	0	1	2	3	4	5	6	7	8
P(x)	0.1	0.2	0.3	0.45	0.6	0.7	0.8	0.9	1.00

It's important to distinguish these and will become important (somewhat) in the binomial case, and very important in the continuous case (in Chapter 5 and beyond).

These table have to follow the same probability rules we talked about in Chapter 3. All the probabilities have to be positive (0 or larger). No probability can be bigger than 1. And all probabilities in the table must add to one. (in the pdf)

You might be asked to check if a table of supposed probability values is a valid probability distribution or not.

What is the probability of being 6? $P(X = 6)$? 0.1
What is probability of being between 2 and 5 inclusive? $\mathrm{P}(2 \leq X \leq 5)$? 0.1+0.15+0.15+0.1=0.5
If a value is missing from the table, use the complement rule to fill in the missing value.

Expected values and standard deviations
Expected values = mean, but in this case, we mean the weighted mean/weighted average

Standard deviation
Trying to average distance to the mean.
Subtract the mean off all the values in the distribution (all the outcomes)
Square the values to make them positive

Multiply by the weights (probabilities)
(in other circumstances, divide by the weights, but here since they are 1, we don't worry about it)
This gives the variance. Take the square root to get the standard deviation.

Expected Value – real world example

Suppose you are attending a fundraising event where there is a raffle that awards prizes to tickets that drawn at the end of the night. Suppose that 400 tickets are sold for $15 each. The top prize is $1000. The second prize is $500. The third prize is $100. And there are 5 4th-place prizes, each are $20. What is the expected value of purchasing a ticket?

outcome	$985	485	85	5	-15
probability	1/400	1/400	1/400	5/400	392/400

Expected values: 985*1/400+485*1/400+85*1/400+5*5/400+(-15)*392/400 = -$10.75

For every ticket purchased, on average, one can expect to lose $10.75.

Binomial Distribution:
The universe of outcomes is divided into two cases. One is called is success, and one is called failure.
The probability of success is labeled p
Probability of failure is the complement: $1 - p \;\; (q)$
Binomial experiment rules n trials (of the experiment), then count the number of successes x or sometimes r.

Suppose I flip a coin 10 times. What is the probability that I will get 5 heads?
$n = 10, x = r = 5, p = 0.5$ (if the coin is fair).

$$\mathrm{P}(X = r|\, binomial) = B(n, p, r) = \binom{n}{r} p^r q^{n-r} = \binom{n}{r} p^r (1-p)^{n-r}$$

$$B(10{,}0.5{,}5) = \binom{10}{5} 0.5^5 (0.5)^5 = 0.246 \ldots$$

More in Excel

Geometric distribution is similar to the binomial except that you are flipping coins until you get 5 successes.
There is a fix probability of success or failure (two outcomes), but the number of flips is varying.

Hypergeometric distribution
This is for a fixed small sample without replacement. Suppose you have 30 marbles, and you want to determine the probability of selecting red marbles but without putting the marbles. Still putting in two outcomes (success=red, and failure is other). Basically, just built on counting rules.

Poisson Distribution
Counts the number of times that an event occurs in a fixed time period.

Related to a continuous distribution called the exponential distribution (probability of time between events).

Next time:
Continue with the binomial distribution.
Talk about expected value and standard deviation
Do more examples of problems: cumulative examples, ranges, etc.
Maybe start continuous distributions.