3/31/2022

Probability, Chapter 3
Terminology and Notation
Independence, Mutually Exclusive
Basic Probability Rules
Contingency (Two-Way) Tables
Tree Diagrams, Venn Diagrams

Probability is the chance that a particular event will occur.
Usually expressed as a ratio (fraction), a percentage or a decimal.
Fraction: the head on a fair coin comes up with probability ½
Percentage: the head on a fair coin comes with up probability 50%
Decimal: the head on a fair coin comes up with probability 0.5
(small decimals will appears as scientific notation; 2.6E-04 is equal to 0.00026; 2.6 is not a probability).

The most basic rules for probabilities is that:
All probabilities are positive (0 or larger, 0 is impossible event)
No probability can be bigger than 1 (1=100% is certainty)
All probabilities for a specific class of events must add up to 100%.

Sample space: the set of all possible outcomes of a particular action (or experiment).
The sample space of a standard die {1,2,3,4,5,6}.
Any one of these outcomes is considered a simple event. {1}
Any combination of these outcomes is considered a compound event. {even outcomes}={2,4,6}

Most of the examples we will talk about will involved simple events that are equally likely.
Equiprobable events. It makes the theoretical probability calculations easier.

Notation:
The probability of obtaining a 1 on a die roll: $P(1), P(X=1)$
The probability of getting a head: $P(H)$

$$P(H)=0.5$$

Theoretical probabilities (classical probabilities) – are generally assumed to from equally likely events. We calculate probabilities by counting up the number of simple events in the (compound) event of interest, and counting up the number of events in the sample space (events that are possible). The probability is just $\frac{Count\ of\ Events}{Count\ of\ Sample\ Space}$
Ex. For a fair die. $P(1)=\frac{1}{6}, P(even)=\frac{3}{6}$

Experimental (empirical) probabilities – it's calculated from actually running the experiment many times and then dividing the number "successes" divided by the number of experiments. For instance, you want to find out if a coin is fair. You toss the coin 5,000 times. And you obtain 2495 heads. Then the empirical probability is $\frac{2495}{5000}$.
You would expect that the more experiments you run, the more similar the probability will be to the theoretical value. (the law of large numbers)

Subjective (personal) probabilities: these are fuzzier, and may involve personal experience, hard to quantify factors, and are difficult to replicate.
Rare events (like the risk of another 9/11 style event), or things that can't be repeated exactly like the sense of likelihood of success for passing a particular exam.

Mutually exclusive:
Mutually exclusive events cannot happen at the same time.
Within an experiment, all simple events are mutually exclusive. When I roll a die, only 1 side can come up on that die at any roll.

Or situations (compound events): if the outcomes are mutually exclusive then the probabilities can add directly.

Probability of getting an even outcome on a die roll:
{2,4,6}/{1,2,3,4,5,6} = 3/6
P(2) = 1/6, P(4)=1/6, P(6)=1/6, P({2,4,6})=1/6+1/6+1/6=3/6

Independent vs. dependent
Independent events do not effect the probability (or our knowledge of the probability) of the other event.
Dependent events are "linked" in that knowing one outcomes means the other is more or less likely than without that knowledge.

Ex. Of an independent event: coin toss and a die roll.
Ex. Of a dependent event: the probability of owning power tools given the information about the person's gender.

When the probabilities of two events are independent and we want them to both occur at the same time (like a H on a coin and a 5 on a die roll), then the probability of the compound event P(H and 5) is the product of the two separate probabilities.

$$P(H\ and\ 5) = P(H) \times P(5) = \frac{1}{2} \times \frac{1}{6} = \frac{1}{12}$$

If we want to calculate an "or" event such as the probability that either a head will come up OR a 5 will come up.

$$P(H\ or\ 5) = P(H) + P(5) - P(H\ and\ 5)$$

The possible outcomes of coin toss are {H, T}
The possible outcomes of the die roll are {1,2,3,4,5,6}

The possible outcomes of both events together:
{(H,1),(H,2),(H,3),(H,4),(H,5),(H,6),(T,1),(T,2),(T,3),(T,4),(T,5),(T,6)}

Events that contain a head: {(H,1),(H,2),(H,3),(H,4),(H,5),(H,6)} – 6 events
Events that contain a 5: {(H,5),(T,5)} – 2 events

(H,5) is in both lists. Only 7 events that contain H or 5, but added separately we get 8 because one event, the combination event is counted twice.

The subtraction in the formula is accounting for that double count.

Complement
The situation where the "event" did not occur.
Notation is non-standard. If the event is labeled A, the complement: A', $\sim A$, $\bar{A}$, and others. Read as "Not A". $P(A') = 1 - P(A)$.

If the event we are interested in is getting a 1 on a die roll, A is the outcome 1.
P(1)=1/6
P(not 1) = P(1')= 1-1/6 = 5/6

Conditional probability:
$P(A|B) =$ the probability of A given B.
If I know that event B has occurred, this is the probability of A.
Dependent probabilities: because the probability of A is different in the general case, than when we know B. P(owning power tools) is different than P(owning power tools | man).

Independent vs. dependent events.
If an event is independent: then $P(A) = P(A|B)$
Or $P(A \text{ and } B) = P(A) \times P(B)$

Dependent events: $P(A) \neq P(A|B)$

$$P(A \text{ and } B) = P(A|B) \times P(B)$$

If we want to know the conditional probability, we can solve the above formula for that term.

$$P(A|B) = \frac{P(A \text{ and } B)}{P(B)}$$

This is called Bayes' Rule.

Contingency Table = Two-Way Table

See Excel:

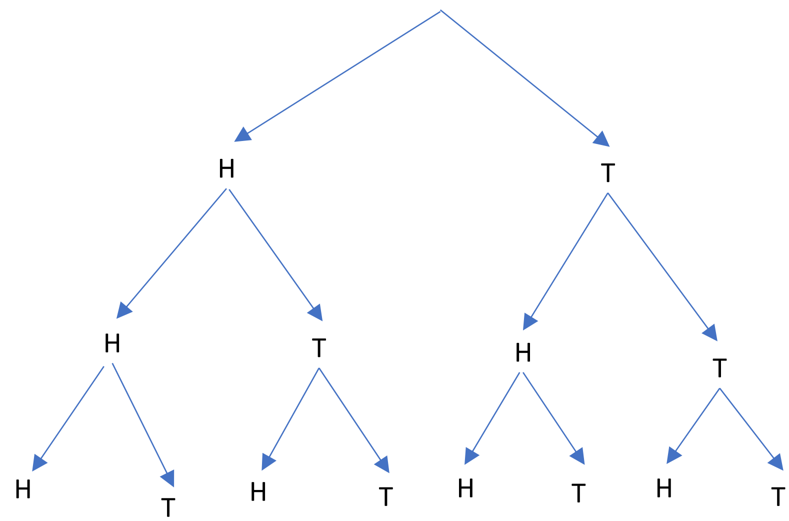
Count of History	Column Labels				
Row Labels	Low	Medium	High	None	Grand Total
Young (<30)	123	35	7	122	287
Middle	69	127	167	145	508
Older (>56)	38	50	81	36	205
Grand Total	230	212	255	303	1000

In Excel we calculate examples of probabilities from continency tables.

Tree Diagrams and Venn Diagrams

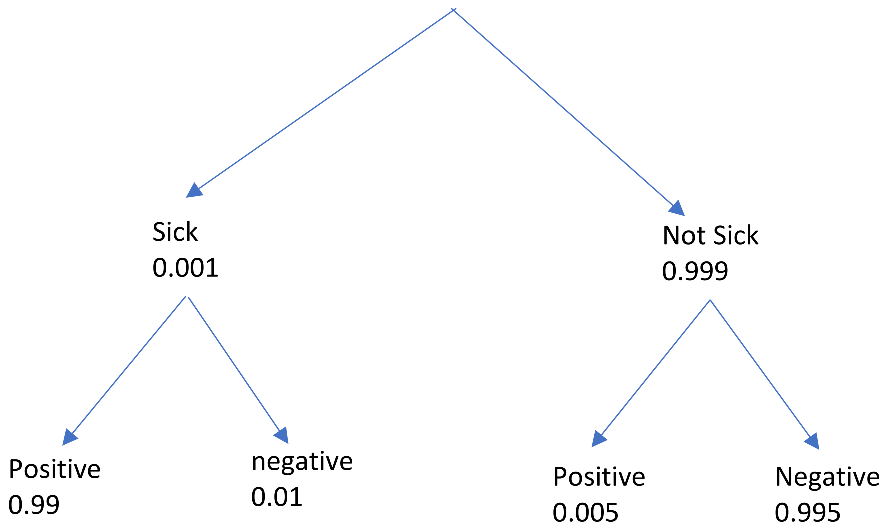
The tree diagram is a way of combining information about multiple events. Either to come up with a list of equally likely events, or to calculate probabilities.

A tree diagram for the outcomes of three coin tosses.



The List of outcomes:
{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT}
8 outcomes
What is the probability of getting 2 heads? 3/8

Bayes' rule example with a tree diagram.
Suppose there is a rare disease that affects only 0.001 (0.1%) of the population. There is a test for the disease that has a 99% accuracy rate (99% of the people with the disease will get a positive test). The percent of people with no disease that receive a false positive is 0.5% (0.005). If you test positive for the disease, what is the chance you actually have the disease?



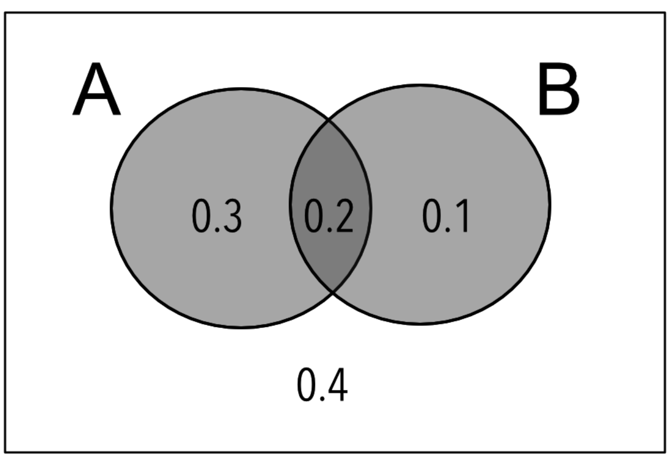
$$P(disease|positve) = \frac{P(disease\ and\ positive)}{P(positive)}$$

$$P(disease\ and\ positive) = P(sick) \times P(sick|positive) = 0.001 \times 0.99$$

$$P(positive) = P(sick\ and\ positive) + P(not\ sick\ and\ positive) = 0.001 \times 0.99 + 0.999 \times 0.005$$

$$P(disease|positive) = \frac{0.001 \times 0.99}{0.001 \times 0.99 + 0.999 \times 0.005} = 0.1654\ \ldots$$

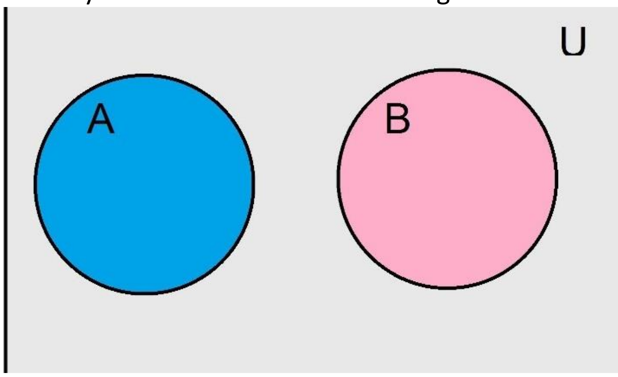
16.5% of the time that a positive test comes back, the person actually has the disease.



P(A and B) = 0.2
P(A) = 0.3+0.2=0.5
P(B)=0.1+0.2=0.3
0.4 is the probability that neither A nor B happened.
P(A or B) = P(A)+P(B)-P(A and B) = 0.3+0.1+0.2=0.6 = 0.5+0.3-0.2=0.6

And = ∩ intersection
Or = ∪ union

Mutually exclusive events in Venn diagrams:



No overlap, means they can't happen at the same time.

Next time, Chapter 4, discrete random variables.