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Invertible Matrix Theorem

The main take-away from section 2.3 is the type of things you can show in a matrix or linear transformation to show that it is invertible. The theorem shows a list of equivalent statements, so that if we know that one is true, then we know all of them are true for a given $n \times n$ matrix.

1. A is invertible
2. A^T is invertible
3. There exists a matrix C such that $CA = I$.
4. There exists a matrix D such that $AD = I$.
5. A is row-equivalent to the identity I.
6. A has a pivot in every column.
7. A has a pivot in every row.
8. $A\vec{x} = \vec{b}$ has a solution for every $\vec{b}$ in R^n.
9. $A\vec{x} = \vec{0}$ has only the trivial solution.
10. The columns of A are linearly independent.
11. The columns of A span R^n.
12. A has n pivots.
13. A as a linear transformation is one-to-one.
14. A as a linear transformation is onto.

There are more equivalencies, but we'll encounter those in later chapters as we introduce more of the terminology. There are more than 20 equivalent statements altogether.

At this point, we have enough to complete the Proof Set #1. Most of the material we've done in the previous chapters. Some pointers:

For #1, row reduce to obtain your condition. For #2, you can do an example since this is technically a counterexample to the equality statement. For #3, a symmetric matrix is a matrix where $a_{ij} = a_{ji}$ for every i, j, or alternatively, $A = A^T$. This second definition is the easiest one to use here since it doesn't depend on the size of the matrix and you don't have to look at individual entries. For #4, make sure you show both parts: a) $A = I$ works, b) non-invertible matrices work. For #5, you are proving definitions. While other problems should be shown on general $m \times n$ or $n \times n$ matrices, I'm only asking you to do these on 2×2 matrices. Look at the operations as defined, and go term by term. You shouldn't need any crazy notation, but be sure to justify steps by using properties of real numbers. Be very careful not to skip steps that seem obvious. Keep asking yourself "but how do I know this?" For #6 and #7, these are back to general size matrices, so you'll need summation notation to do these. You can find the proofs online, but be sure to explain them in your own words. You can start with a smaller matrix to get familiar with what is happening, but still be sure to generalize it. If you've had Calc II, I think #8 is the easiest of the optional proofs, but hardly anyone ever does it.

Chapter 3 is working with determinants, calculating them, and Cramer's Rule. There are two basic methods for finding determinants: The co-factor method and the row-reducing method.

The co-factor method breaks a large matrix up into smaller matrices until we get down to 2×2 matrices, since the determinant of 2×2 matrices is $ad - bc$, just like our denominator from the inverse matrix formula.

Consider the 4×4 matrix shown. Find the determinant.

$$\begin{vmatrix} 1 & 3 & 5 & 2 \\ 1 & 1 & -4 & 1 \\ 3 & -2 & 0 & 0 \\ 0 & 1 & -1 & 1 \end{vmatrix}$$

A note about notation: the |A| notation just means "take the determinant of A", and is equivalent to $\det(A)$.

When we use the cofactor method, we expand upon any row or column using the following sign conventions:

$$\begin{vmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{vmatrix}$$

Basically, start with a + in the top left, and flip signs each time you move over a row or a column.

We'll expand on the top row first. Then we'll come back and show that we can chose another row or column and get the same answer.

$$1\begin{vmatrix} 1 & -4 & 1 \\ -2 & 0 & 0 \\ 1 & -1 & 1 \end{vmatrix} - (3)\begin{vmatrix} 1 & -4 & 1 \\ 3 & 0 & 0 \\ 0 & -1 & 1 \end{vmatrix} + 5\begin{vmatrix} 1 & 1 & 1 \\ 3 & -2 & 0 \\ 0 & 1 & 1 \end{vmatrix} - (2)\begin{vmatrix} 1 & 1 & -4 \\ 3 & -2 & 0 \\ 0 & 1 & -1 \end{vmatrix}$$

The smaller matrix multiplying the co-factor is the rest of the A matrix after the row and column that the co-factor is sitting in is deleted. The row and column containing the 5, is highlighted in the original matrix, the matrix in the expansion contains all the un-highlighted entries.

There are a number of ways to find the 3×3 matrices. Some of them only work on matrices of that size and don't work on other sizes, so we'll just keep going.

$$\begin{vmatrix} 1 & -4 & 1 \\ -2 & 0 & 0 \\ 1 & -1 & 1 \end{vmatrix} = 1\begin{vmatrix} 0 & 0 \\ -1 & 1 \end{vmatrix} - (-4)\begin{vmatrix} -2 & 0 \\ 1 & 1 \end{vmatrix} + 1\begin{vmatrix} -2 & 0 \\ 1 & -1 \end{vmatrix} =$$
$$1(0) + 4(-2-0) + (2-0) = 0 - 8 + 2 = -6$$

$$\begin{vmatrix} 1 & -4 & 1 \\ 3 & 0 & 0 \\ 0 & -1 & 1 \end{vmatrix} = 1\begin{vmatrix} 0 & 0 \\ -1 & 1 \end{vmatrix} - (-4)\begin{vmatrix} 3 & 0 \\ 0 & 1 \end{vmatrix} + 1\begin{vmatrix} 3 & 0 \\ 0 & -1 \end{vmatrix} =$$
$$1(0) + 4(3-0) + (-3-0) = 0 + 12 - 3 = 9$$

$$\begin{vmatrix} 1 & 1 & 1 \\ 3 & -2 & 0 \\ 0 & 1 & 1 \end{vmatrix} = 1\begin{vmatrix} -2 & 0 \\ 1 & 1 \end{vmatrix} - (1)\begin{vmatrix} 3 & 0 \\ 0 & 1 \end{vmatrix} + 1\begin{vmatrix} 3 & -2 \\ 0 & 1 \end{vmatrix} =$$
$$1(-2-0) - 1(3-0) + (3-0) = -2 - 3 + 3 = -2$$

$$\begin{vmatrix} 1 & 1 & -4 \\ 3 & -2 & 0 \\ 0 & 1 & -1 \end{vmatrix} = 1\begin{vmatrix} -2 & 0 \\ 1 & -1 \end{vmatrix} - (1)\begin{vmatrix} 3 & 0 \\ 0 & -1 \end{vmatrix} + (-4)\begin{vmatrix} 3 & -2 \\ 0 & 1 \end{vmatrix} =$$
$$1(-2-0) - 1(3-0) - 4(3-0) = -2 - 3 - 12 = -7$$

Putting all this back into our original expression

$$1\begin{vmatrix} 1 & -4 & 1 \\ -2 & 0 & 0 \\ 1 & -1 & 1 \end{vmatrix} - (3)\begin{vmatrix} 1 & -4 & 1 \\ 3 & 0 & 0 \\ 0 & -1 & 1 \end{vmatrix} + 5\begin{vmatrix} 1 & 1 & 1 \\ 3 & -2 & 0 \\ 0 & 1 & 1 \end{vmatrix} - (2)\begin{vmatrix} 1 & 1 & -4 \\ 3 & -2 & 0 \\ 0 & 1 & 1 \end{vmatrix} =$$
$$1(-6) - 3(9) + 5(-2) - 2(-7) = -29$$

This is kinda tedious, but you can expand on any row or column to take advantage of zeros that appear in the matrix. For instance, what if we used the third row instead?

$$\begin{vmatrix} 1 & 3 & 5 & 2 \\ 1 & 1 & -4 & 1 \\ 3 & -2 & 0 & 0 \\ 0 & 1 & -1 & 1 \end{vmatrix} = 3\begin{vmatrix} 3 & 5 & 2 \\ 1 & -4 & 1 \\ 1 & -1 & 1 \end{vmatrix} - (-2)\begin{vmatrix} 1 & 5 & 2 \\ 1 & -4 & 1 \\ 0 & -1 & 1 \end{vmatrix} + 0\begin{vmatrix} 1 & 3 & 2 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix} - 0\begin{vmatrix} 1 & 3 & 5 \\ 1 & 1 & -4 \\ 0 & 1 & -1 \end{vmatrix}$$

But, you can see that the deteminant of the last two matrices doesn't matter, since they are multiplied by 0. So we just have to worry about the two others.

$$\begin{vmatrix} 3 & 5 & 2 \\ 1 & -4 & 1 \\ 1 & -1 & 1 \end{vmatrix} = 3\begin{vmatrix} -4 & 1 \\ -1 & 1 \end{vmatrix} - 5\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} + 2\begin{vmatrix} 1 & -4 \\ 1 & -1 \end{vmatrix} =$$
$$3(-4+1) - 5(1-1) + 2(-1+4) = 3(-3) - 5(0) + 2(3) = -9 - 0 + 6 = -3$$

$$\begin{vmatrix} 1 & 5 & 2 \\ 1 & -4 & 1 \\ 0 & -1 & 1 \end{vmatrix} = 1\begin{vmatrix} -4 & 1 \\ -1 & 1 \end{vmatrix} - 5\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} + 2\begin{vmatrix} 1 & -4 \\ 0 & -1 \end{vmatrix} =$$
$$(-4+1) - 5(1-0) + 2(-2-0) = -3 - 5 - 4 = -10$$

So, we end up with:

$$3\begin{vmatrix} 3 & 5 & 2 \\ 1 & -4 & 1 \\ 1 & -1 & 1 \end{vmatrix} - (-2)\begin{vmatrix} 1 & 5 & 2 \\ 1 & -4 & 1 \\ 0 & -1 & 1 \end{vmatrix} + 0\begin{vmatrix} 1 & 3 & 2 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix} - 0\begin{vmatrix} 1 & 3 & 5 \\ 1 & 1 & -4 \\ 0 & 1 & -1 \end{vmatrix} =$$
$$3(-3) + 2(-10) + 0 + 0 = -9 - 20 = -29$$

So, we got the same answer, with a lot less work.

For the row-reducing method, we reduce the matrix using row operations in order to take advantage of the zeros. The textbook reduces to the identity, but we can reduce the size as we go.

Some row operations change the determinant. If we use them, then we have to keep track of the changes. Switches rows changes the sign of the determinant. Multiplying a row by a constant, changes the determinant by that constant. So both $3R_1 \to R_1$ and $3R_1 + R_2 \to R_1$ changes the determinant by 3. But $3R_1 + R_2 \to R_2$ does not.

$$\begin{vmatrix} 1 & 3 & 5 & 2 \\ 1 & 1 & -4 & 1 \\ 3 & -2 & 0 & 0 \\ 0 & 1 & -1 & 1 \end{vmatrix}$$

Use the 1 in the first row to get rid of the 1 and 3 below it.

$$-R_1 + R_2 \to R_2$$
$$-3R_1 + R_3 \to R_3$$

(These don't induce any changes.)

$$\begin{vmatrix} 1 & 3 & 5 & 2 \\ 0 & -2 & -9 & -1 \\ 0 & -11 & -15 & -6 \\ 0 & 1 & -1 & 1 \end{vmatrix}$$

Then we can reduce the size of the matrix by expanding on the first column.

$$\begin{vmatrix} 1 & 3 & 5 & 2 \\ 0 & -2 & -9 & -1 \\ 0 & -11 & -15 & -6 \\ 0 & 1 & -1 & 1 \end{vmatrix} = 1 \begin{vmatrix} -2 & -9 & -1 \\ -11 & -15 & -6 \\ 1 & -1 & 1 \end{vmatrix}$$

If we $R_1 \leftrightarrow R_3$ we'll change the sign.

$$\begin{vmatrix} 1 & -1 & 1 \\ -11 & -15 & -6 \\ -2 & -9 & -1 \end{vmatrix}$$

Then we can get rid of the -11 and -2.

$$11R_1 + R_2 \to R_2$$
$$2R_2 + R_3 \to R_3$$

$$\begin{vmatrix} 1 & -1 & 1 \\ 0 & -26 & 5 \\ 0 & -11 & 1 \end{vmatrix}$$

Expanding on the first column:

$$\begin{vmatrix} 1 & -1 & 1 \\ 0 & -26 & 5 \\ 0 & -11 & 1 \end{vmatrix} = 1 \begin{vmatrix} -26 & 5 \\ -11 & 1 \end{vmatrix} = -26 + 55 = 29$$

But remember, we changed the sign when we switched rows, so the original matrix determinant was $-(29) = -29$.

You'll need to be able to do both methods for the exam. In general, when you have a choice, the cofactor method works best when there are a lot of 0's, and row-reducing works best as the matrices get larger, and if there aren't a lot of zeros.