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Systems of linear equations

Linear algebra is fundamentally about analyzing systems of linear equations, or things that can be expressed as systems of linear equations. This includes a surprising number of situations, but we'll start with the basics.

In pre-calc we saw systems like this:

$$\begin{cases} 4x - 3y + z = -10 \\ 2x + y + 3z = 0 \\ -x + 2y - 5z = 17 \end{cases}$$

Now, they have this form since we want to generalize to more variables:

$$\begin{cases} 4x_1 - 3x_2 + x_3 = -10 \\ 2x_1 + x_2 + 3x_3 = 0 \\ -x_1 + 2x_2 - 5x_3 = 17 \end{cases}$$

If we convert this to a single matrix, called an augmented matrix, we use one column for the coefficients of each variable, one for the constants, and then each row corresponds to the equations. The positions in the matrix signify what they mean.

$$\left[\begin{array}{ccc|c} 4 & -3 & 1 & -10 \\ 2 & 1 & 3 & 0 \\ -1 & 2 & -5 & 17 \end{array}\right]$$

If we want to solve the equation, we can use this format.

Row operations on a matrix are essentially equivalent to doing elimination by addition. We can do three things: 1) swap rows, 2) add rows together, 3) multiply a row by a scalar. We can also do the last two operations together.

The notation for swapping rows is:
Exchange Row 1 and Row 2: $R_1 \leftrightarrow R_2$

To add rows, we also have to say where to put the result back into:
Add Row 2 and Row 3 and put back in Row 3: $R_2 + R_3 \to R_3$

To scale one row:
Multiply Row 2 by ½: $\frac{1}{2}R_2 \to R_2$

To combine the last two:
We can Add -2 times Row 1 to Row 3, and put back in Row 3: $-2R_1 + R_3 \to R_3$

We apply these operations to get an augmented matrix in (row) echelon form or (row) reduced echelon form.

(Row) Echelon Form ideally for a unique solution looks like this:

$$\left[\begin{array}{ccc|c} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \end{array}\right]$$

The * mean they can be any number, the others are specific. The locations of the 1's are called pivots. They mark the "stair" from which "echelon" form gets its name. This form is not unique (the * values depend on the operations performed), but the pivot positions and last row will be the same.

(Row) Reduced Echelon Form take this further:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \end{array}\right]$$

The remaining * are now the solution to the system since every equation just has one variable equal to a constant.

When doing this work by hand, it's generally more efficient for people to stop at echelon form and back solve. When using technology, and for some types of systems (when there is no unique solution), we should go to reduced echelon form.

Let's solve our system using this process, called Gauss-Jordan elimination.

First, we create a pivot in the top position a_{11} (first row, first column), and then use that 1 to eliminate all the non-zero entries below it.

$$\left[\begin{array}{ccc|c} 4 & -3 & 1 & -10 \\ 2 & 1 & 3 & 0 \\ -1 & 2 & -5 & 17 \end{array}\right]$$

I'm going to move Row 3 to Row 1, and then multiply by -1. Then, use that to eliminate the 2 and 4.

$$R_1 \leftrightarrow R_3$$

$$\left[\begin{array}{ccc|c} -1 & 2 & -5 & 17 \\ 2 & 1 & 3 & 0 \\ 4 & -3 & 1 & -10 \end{array}\right]$$

$$-1R_1 \to R_1$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 5 & -17 \\ 2 & 1 & 3 & 0 \\ 4 & -3 & 1 & -10 \end{array}\right]$$

$$-2R_1 + R_2 \to R_2$$
$$-4R_1 + R_3 \to R_3$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 5 & -17 \\ 0 & 5 & -7 & 34 \\ 0 & 5 & -19 & 58 \end{array}\right]$$

Then we move on to position a_{22} (row 2, column 2) and turn that into 1, and then use that to eliminate the 5 below it. As people, we might find a more efficient way to proceed, but for computers, we need them to use a methodical process that can be applied without thought. Still, we should end up in basically the same place.

$$\frac{1}{5}R_2 \to R_2$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 5 & -17 \\ 0 & 1 & -\frac{7}{5} & \frac{34}{5} \\ 0 & 5 & -19 & 58 \end{array}\right]$$

$$-5R_2 + R_3 \to R_3$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 5 & -17 \\ 0 & 1 & -\frac{7}{5} & \frac{34}{5} \\ 0 & 0 & -12 & 24 \end{array}\right]$$

Finally, make the -12 into a 1.

$$-\frac{1}{12}R_3 \to R_3$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 5 & -17 \\ 0 & 1 & -\frac{7}{5} & \frac{34}{5} \\ 0 & 0 & 1 & -2 \end{array}\right]$$

This is echelon form.
The system now looks like this:

$$\begin{cases} x_1 - 2x_2 + 5x_3 = -17 \\ x_2 - \frac{7}{5}x_3 = \frac{34}{5} \\ x_3 = -2 \end{cases}$$

So, last variable is -2. We now can backsolve to get a solution.

$$x_2 - \frac{7}{5}(-2) = \frac{34}{5}$$
$$x_2 = 4$$

$$x_1 - 2(4) + 5(-2) = -17$$
$$x_1 = 1$$

So, in vector form the solution is $\vec{x} = \begin{bmatrix} 1 \\ 4 \\ -2 \end{bmatrix}$.

Or, we can continue applying row reducing to get to reduced echelon form.

$$\left[\begin{array}{ccc|c} 1 & -2 & 5 & -17 \\ 0 & 1 & -\frac{7}{5} & \frac{34}{5} \\ 0 & 0 & 1 & -2 \end{array}\right]$$

Multiply Row 3 by $\frac{7}{5}$ and add to Row 2. And -5 times Row 3 added to Row 1.

$$\frac{7}{5}R_3 + R_2 \to R_2$$
$$-5R_3 + R_1 \to R_1$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 0 & -7 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

And finally 2 times Row 2 adding to Row 1.

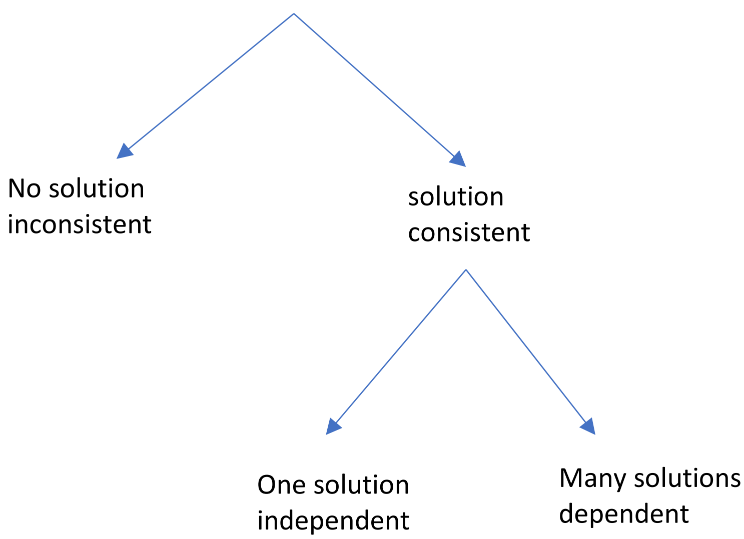
$$2R_2 + R_1 \rightarrow R_1$$
$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

And you can see that the final column is the same as our solution.

We call this system consistent since a solution exists, and independent since there is a unique solution.

If the system has infinite numbers of solutions (fewer pivots than variables), then the system is dependent.

If there is no solution (the last line of the matrix translates to 0 = a number), then we say it's inconsistent.



We can also write our systems in two other formats that will be useful later on.

As a vector equation.

$$\begin{bmatrix} 4 \\ 2 \\ -1 \end{bmatrix} x_1 + \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix} x_2 + \begin{bmatrix} 1 \\ 3 \\ -5 \end{bmatrix} x_3 = \begin{bmatrix} -10 \\ 0 \\ 17 \end{bmatrix}$$

As a matrix equation.

$$\begin{bmatrix} 4 & -3 & 1 \\ 2 & 1 & 3 \\ -1 & 2 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -10 \\ 0 \\ 17 \end{bmatrix}$$

The vector equation will be conceptually important when we work with coordinate systems.
The matrix equation format will be important when we talk about linear transformations and solve systems with inverse matrices and Cramer's rule.

When we have too few variables to solve for all them (either too few equations or one disappears as we solve), then we have a dependent solution, and we have to express it parametrically.

Consider the system:

$$\begin{cases} 2x + y - 3z = 6 \\ 3x + 3y - 7z = 7 \\ x - y + z = 5 \end{cases}$$

We switch Row 1 and Row 3 so we don't introduce fractions.

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 5 \\ 3 & 3 & -7 & 7 \\ 2 & 1 & -3 & 6 \end{array}\right]$$

Then multiply -3 by Row 1, and add to Row 2. Then multiply Row 1 by -2 and add to Row 3.

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 5 \\ 0 & 6 & -10 & -8 \\ 0 & 3 & -5 & -4 \end{array}\right]$$

We can see that if we multiply Row 3 by -2 and add to Row 2 we get all 0's

$$-2R_3 + R_2 \to R_3$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 5 \\ 0 & 6 & -10 & -8 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

This means the system is dependent, so we have to go all the way to reduced echelon form to solve it.
Divide Row 2 by 6.

$$1/6R_2 \to R_2$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 5 \\ 0 & 1 & -\frac{5}{3} & -\frac{4}{3} \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Then add Row 2 to Row 1.

$$\left[\begin{array}{ccc|c} 1 & 0 & -\frac{2}{3} & \frac{11}{3} \\ 0 & 1 & -\frac{5}{3} & -\frac{4}{3} \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Our system is now:

$$\begin{cases} x_1 - \frac{2}{3}x_3 = \frac{11}{3} \\ x_2 - \frac{5}{3}x_3 = -\frac{4}{3} \end{cases}$$

To solve the system, move the x_3 terms to the right side and add an equation to complete the solution that is a tautology: $x_3 = x_3$

$$\begin{cases} x_1 = \frac{2}{3}x_3 + \frac{11}{3} \\ x_2 = \frac{5}{3}x_3 - \frac{4}{3} \\ x_3 = x_3 \end{cases}$$

The left side is $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, and the right side are the coefficients of x_3 and constants. We write these as their own vectors.

$$\begin{bmatrix} \frac{2}{3} \\ \frac{5}{3} \\ 1 \end{bmatrix} x_3 + \begin{bmatrix} \frac{11}{3} \\ -\frac{4}{3} \\ 0 \end{bmatrix}$$

We can replace x_3 with a parameter that we can use to simplify the express. If we choose $x_3 = 3t$ this will reduce to

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 3 \end{bmatrix} t + \begin{bmatrix} \frac{11}{3} \\ -\frac{4}{3} \\ 0 \end{bmatrix}$$

We can't eliminate the fractions in the constant vector though. But now, if we want to find any solution to the system, we choose a value for t and get a solution that works.

We can think of this as a line shifted off the origin: $\vec{x} = \vec{v}t + \vec{p}$.

The terms independent and dependent come from the vector form of the equation. If we find a pivot in every column, the vectors are independent (we can't solve one of them from the others), and so the solution is unique. If the vectors are dependent, then, as with this last problem, we are missing a pivot in the columns, and the solution is not unique.

There are handouts posted in the course on calculator operations for these things. There will be problems you'll need to do by hand (3 variables or less) but larger systems, it will help to be able to use the calculator. There is also a handout on dependent systems that we'll return to as we develop more concepts.

Some extra online resources for this material:
Solving systems in a matrix form: https://www.youtube.com/watch?v=AUqeb9Z3y3k
Vector form of system: http://www.dankalman.net/mwweb/microworlds/lineqns/mateqn.html
Matrix form of an equation: https://www.youtube.com/watch?v=uFzQhPazVYc
Dependent Systems: https://www.youtube.com/watch?v=BeGV5Wwat9g