MTH 266, Exam #1, Part I, Spring 2019 Name KEY

Instructions: Show all work. Give exact answers unless specifically asked to round. If you do not show work, problems will be graded as "all or nothing" for the answer only; partial credit will not be possible and any credit awarded for the work will not be available. On this portion of the exam, you may **NOT** use a calculator.

1. Given the system of equations $\begin{cases} x_1 \quad - 5x_3 = 1 \\ 2x_1 + 4x_2 \quad = 10 \\ -3x_2 - 6x_3 = -3 \end{cases}$, write the *system* as:

   a. An augmented matrix (4 points)

$$\left[\begin{array}{ccc|c} 1 & 0 & -5 & 1 \\ 2 & 4 & 0 & 10 \\ 0 & -3 & -6 & -3 \end{array}\right]$$

   b. A vector equation (4 points)

$$\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} x_1 + \begin{bmatrix} 0 \\ 4 \\ -3 \end{bmatrix} x_2 + \begin{bmatrix} -5 \\ 0 \\ -6 \end{bmatrix} x_3 = \begin{bmatrix} 1 \\ 10 \\ -3 \end{bmatrix}$$

   c. A matrix equation. (4 points)

$$\begin{bmatrix} 1 & 0 & -5 \\ 2 & 4 & 0 \\ 0 & -3 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 10 \\ -3 \end{bmatrix}$$

   d. Solve the system using the augmented matrix and row operations. State whether the solution of the system is consistent or inconsistent. If the system is consistent, state whether it is independent or dependent. Write an independent solution in vector form; write a dependent solution in parametric form. (8 points)

$-2R_1 + R_2 \to R_2$ $\left[\begin{array}{ccc|c} 1 & 0 & -5 & 1 \\ 0 & 4 & 10 & 8 \\ 0 & -3 & -6 & -3 \end{array}\right]$ $\frac{1}{4}R_2 \to R_2$, $\frac{1}{3}R_3 \to R_3$ $\left[\begin{array}{ccc|c} 1 & 0 & -5 & 1 \\ 0 & 1 & 5/2 & 2 \\ 0 & -1 & -2 & -1 \end{array}\right]$

$R_2 + R_3 \to R_3$ $\left[\begin{array}{ccc|c} 1 & 0 & -5 & 1 \\ 0 & 1 & 5/2 & 2 \\ 0 & 0 & 1/2 & 1 \end{array}\right]$ $2R_3 \to R_3$ $\left[\begin{array}{ccc|c} 1 & 0 & -5 & 1 \\ 0 & 1 & 5/2 & 2 \\ 0 & 0 & 1 & 2 \end{array}\right]$

$5R_3 + R_1 \to R_1$, $-5/2 R_3 + R_2 \to R_2$ $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 11 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 2 \end{array}\right]$ $\vec{x} = \begin{bmatrix} 11 \\ -3 \\ 2 \end{bmatrix}$

Consistent
independent

2. Given $A=\begin{bmatrix} 3 & 7 \\ -2 & 1 \end{bmatrix}$, find A^{-1}. (8 points)

$$\frac{1}{3+14}\begin{bmatrix} 1 & -7 \\ 2 & 3 \end{bmatrix} = \frac{1}{17}\begin{bmatrix} 1 & -7 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 1/17 & -7/17 \\ 2/17 & 3/17 \end{bmatrix}.$$

3. Given $A=\begin{bmatrix} 3 & 0 \\ -1 & 5 \end{bmatrix}$, $B=\begin{bmatrix} 0 & 5 & -1 \\ 4 & -2 & 0 \end{bmatrix}$, $C=\begin{bmatrix} 9 \\ -8 \\ 4 \end{bmatrix}$, compute the following, if possible. If the combination is not possible, briefly explain why. (6 points each)

a) AB

$$\begin{bmatrix} 3*0+0*4 & 3*5+0*-2 & 3*-1+0*0 \\ -1*0+5*4 & -1*5+5*-2 & -1*-1+5*0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 15 & -3 \\ 20 & -15 & 1 \end{bmatrix}$$

b) CB

not defined

$(3\times 1)\,(2\times 3)$

no match

BC is defined

c) B^T

$$\begin{bmatrix} 0 & 4 \\ 5 & -2 \\ -1 & 0 \end{bmatrix}$$

d) $3I_2 + A$

$$\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ -1 & 5 \end{bmatrix} =$$

$$\begin{bmatrix} 6 & 0 \\ -1 & 8 \end{bmatrix}$$

4. Use matrix multiplication to determine if $\vec{x} = \begin{bmatrix}3\\5\\-2\end{bmatrix}$ is a solution to the system

$$\begin{cases} x_1 \quad\quad\quad - 3x_3 = 9 \\ 2x_1 - 2x_2 - 7x_3 = 10 \\ \quad\quad - x_2 - 5x_3 = 6 \end{cases}$$. (8 points)

$$\begin{bmatrix}1&0&-3\\2&-2&-7\\0&-1&-5\end{bmatrix}\begin{bmatrix}3\\5\\-2\end{bmatrix} = \begin{bmatrix}3*1+0*5+-3*-2\\2*3+-2*5+-7*-2\\0*3+-1*5+-5*-2\end{bmatrix} = \begin{bmatrix}3+6\\6-10+14\\-5+10\end{bmatrix} = \begin{bmatrix}9\\10\\5\end{bmatrix}$$

no, it is not a solution since the product is not $\begin{bmatrix}9\\10\\6\end{bmatrix}$

5. Graph the vectors $\vec{u} = \begin{bmatrix}1\\0\end{bmatrix}$ *and* $\vec{v} = \begin{bmatrix}-2\\3\end{bmatrix}$ and label which vector is which on the graph. On the same graph also plot the following and label each part clearly: (16 points)

a. $\vec{u} + \vec{v}$
b. $3\vec{u} - 2\vec{v}$
c. For $A = \begin{bmatrix}1&2\\0&1\end{bmatrix}$, plot $A\vec{v}$
d. Rotate $\vec{u}$ through a counterclockwise angle of $\frac{3\pi}{4}$

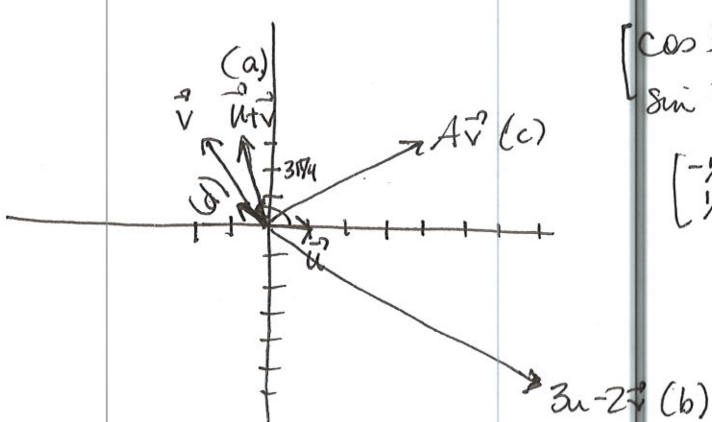
$\vec{u}+\vec{v} = \begin{bmatrix}-1\\3\end{bmatrix}$

$3\vec{u} - 2\vec{v} = \begin{bmatrix}3\\0\end{bmatrix} + \begin{bmatrix}4\\-6\end{bmatrix} = \begin{bmatrix}7\\-6\end{bmatrix}$

$\begin{bmatrix}1&2\\0&1\end{bmatrix}\begin{bmatrix}-2\\3\end{bmatrix} = \begin{bmatrix}-2+6\\0+3\end{bmatrix} = \begin{bmatrix}4\\3\end{bmatrix} = A\vec{v}$

$\begin{bmatrix}\cos\frac{3\pi}{4} & -\sin\frac{3\pi}{4}\\ \sin 3\pi/4 & \cos 3\pi/4\end{bmatrix} = \begin{bmatrix}-1/\sqrt{2} & -1/\sqrt{2}\\ 1/\sqrt{2} & -1/\sqrt{2}\end{bmatrix}$

$\begin{bmatrix}-1/\sqrt{2} & -1/\sqrt{2}\\ 1/\sqrt{2} & -1/\sqrt{2}\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}-1/\sqrt{2}\\1/\sqrt{2}\end{bmatrix}$ (d)



6. Determine if each statement is True or False. (3 points each)

a. T F Every linear transformation on a finite vector space is a matrix transformation and every matrix transformation is a linear transformation.

b. T F If *A* is a $m \times n$ matrix that has *n* pivot columns, then the equation $A\mathbf{x} = \mathbf{b}$ is unique for all **b** in $\Re^m$.

c. T F If A is a 3x3 matrix, then the transformation $\vec{x} \mapsto A\vec{x}$ must be one-to-one and onto.

needs 3 pivots

d. T F Matrix multiplication is associative.

e. T F The result of multiplication between a 2x3 matrix and a 3x2 matrix results in a 3x3 matrix.

2x2 is the result

f. T F If a system of equations has a free variable then it has a unique solution.

no free vars is unique

g. T F If A is a 2x2 matrix for a projection transformation, it is not invertible.

h. T F The equation $\vec{x} = \vec{p} + t\vec{v}$ describes a line through $\vec{p}$ parallel to $\vec{v}$.

i. T F $\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 5 \\ -3 \\ 7 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 11 \end{bmatrix}$ form a linearly independent set.

too many vectors

j. T F The mapping defined by the differential operator $\frac{d}{dx}$ is a linear transformation.

k. T F The pivot positions in a matrix depend on whether row interchanges take place. does not depend

l. T F The linear transformation given by A= $\begin{bmatrix} 1 & 0 & 1 & 0 & -2 \\ 0 & 1 & 3 & 0 & 1 \\ 0 & 0 & 0 & 1 & 5 \end{bmatrix}$ is onto.

3 pivots
one in each row

MTH 266, Exam #1, Part II, Spring 2019 Name KEY

Instructions: Show all work. Give exact answers unless specifically asked to round. All complex numbers should be stated in standard form, and all complex fractions should be simplified. If you do not show work, problems will be graded as "all or nothing" for the answer only; partial credit will not be possible and any credit awarded for the work will not be available. On this portion of the exam, you *may* use a calculator to perform elementary matrix operations. Support your answers with work (reproduce the reduced matrices from your calculator) or other justification for full credit.

7. Find the general solution to the system $\begin{cases} x_1 - 2x_2 + 4x_3 + 5x_4 = 2 \\ -x_1 + x_2 - 3x_3 + x_4 = 7 \end{cases}$. State whether the solution of the system is consistent or inconsistent. If the system is consistent, state whether it is independent or dependent. Write an independent solution in vector form; write a dependent solution in parametric form. Circle the pivots of the reduced matrix. (10 points)

$$\text{rref} \Rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 2 & -7 & -16 \\ 0 & 1 & -1 & -6 & -9 \end{array}\right]$$

Consistent
dependent

$$\begin{aligned} x_1 + 2x_3 - 7x_4 &= -16 \\ x_2 - x_3 - 6x_4 &= -9 \end{aligned} \rightarrow \begin{aligned} x_1 &= -2x_3 + 7x_4 - 16 \\ x_2 &= x_3 + 6x_4 - 9 \\ x_3 &= x_3 \\ x_4 &= x_4 \end{aligned}$$

$$\vec{x} = \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix} x_3 + \begin{bmatrix} 7 \\ 6 \\ 0 \\ 1 \end{bmatrix} x_4 + \begin{bmatrix} -16 \\ -9 \\ 0 \\ 0 \end{bmatrix}$$

8. Determine if $\mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 2 \\ -6 \end{bmatrix}$ is in the span of the columns of $A = \begin{bmatrix} -1 & 2 & 1 & 5 \\ 3 & 0 & 2 & 2 \\ -4 & 2 & 5 & 9 \\ 1 & 3 & 0 & 6 \end{bmatrix}$. If it is, write **b** as a linear combination of the columns of *A*; if not, explain why it is not, and give an example of a vector that *is* in the span. (8 points)

$$\left[\begin{array}{cccc|c} -1 & 2 & 1 & 5 & 1 \\ 3 & 0 & 2 & 2 & 0 \\ -4 & 2 & 5 & 9 & 2 \\ 1 & 3 & 0 & 6 & -6 \end{array}\right] \Rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

no $\vec{b}$ is not in the span since the system represented by the augmented matrix is inconsistent

9. Let A = $\begin{bmatrix} 1 & 3 & 0 & 9 & 8 \\ -1 & -4 & 2 & -7 & 0 \\ 0 & 6 & 1 & -1 & 0 \\ 2 & 2 & 3 & 0 & 1 \\ 7 & 0 & -5 & -3 & -1 \\ 1 & 0 & 11 & 2 & 4 \end{bmatrix}$

a. Determine if the columns of A form a linearly independent or dependent set and justify your answer. (6 points)

yes. they are independent
the reduced echelon form of the matrix has a pivot in every column

b. Determine if the columns of A span R^6. Justify your answer. (6 points)

they do not, since there are only 5 pivots not 6

c. Use the information obtained in parts a and b to determine if the linear transformation $T: \vec{x} \in R^5 \mapsto A\vec{x} \in R^6$ is one-to-one or onto. Justify your answer. (8 points)

One-to-one only
(not onto)

10. **Use an inverse matrix** to solve $\begin{cases} x_1 \qquad\quad - 2x_3 = 1 \\ -3x_1 + x_2 + 4x_3 = -5 \\ 2x_1 - 3x_2 + 4x_3 = 8 \end{cases}$. Give the inverse matrix used. (10 points)

$A = \begin{bmatrix} 1 & 0 & -2 \\ -3 & 1 & 4 \\ 2 & -3 & 4 \end{bmatrix}$ $\quad A^{-1} = \begin{bmatrix} 8 & 3 & 1 \\ 10 & 4 & 1 \\ 7/2 & 3/2 & 1/2 \end{bmatrix} \begin{bmatrix} 1 \\ -5 \\ 8 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$

11. Not all linear transformations can be written as matrices, such as the derivative operator, because they operate on an infinite dimensional vector space (the set of all possible functions); however, if we limit such operators to a finite dimensional space, we can write the linear operator as a matrix. Consider the space P_3 defined as the set of all polynomials of degree 3 or less. These polynomials of the form $p(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$ can be written as a 4-dimensional vector, since all their components can be determined by a set of 4 constants.

a. Write the general polynomial p(t) above as a vector in R^4. (4 points)

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

b. Take the derivative of p(t) and write the resulting vector (now in $P_2 \sim R^3$). (4 points)

$$p(t)' = a_1 + 2a_2 t + 3a_3 t^2 \qquad \begin{bmatrix} a_1 \\ 2a_2 \\ 3a_3 \end{bmatrix}$$

c. Create a matrix linear transformation capable of transforming the vector in part a to the vector in part b, i.e. find A such that $\vec{p} \mapsto A\vec{p} = \vec{p}'$. (8 points)

$$T\left(\begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 \\ 2x_2 \\ 3x_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

12. The invertible matrix theorem states that several statements are equivalent to matrix A being invertible. Name 6 of these equivalent statements (so far there are 11 to choose from). (12 points)

answers will vary

1) reduces to the identity (row equivalent to)
2) $A\vec{x} = \vec{0}$ has only trivial solution
3) $\exists$ C such that $CA = I$
4) $\exists$ D such that $AD = I$
5) A is one-to-one
6) vectors of A span $\mathbb{R}^n$ etc.

13. Prove that the transformation T defined by the $T: \vec{x} \mapsto A\vec{x}$ for the matrix $A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}$ is linear using the definition. (12 points)

$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ $A\vec{x} = \begin{bmatrix} x_2 \\ x_1 + x_3 \\ x_2 + 2x_3 \end{bmatrix}$ $A\vec{y} = \begin{bmatrix} y_2 \\ y_1 + y_3 \\ y_2 + 2y_3 \end{bmatrix}$

$\vec{x} + \vec{y} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 \end{bmatrix}$ $A(\vec{x} + \vec{y}) = \begin{bmatrix} x_2 + y_2 \\ x_1 + y_1 + x_3 + y_3 \\ x_2 + y_2 + 2(x_3 + y_3) \end{bmatrix} = A\vec{x} + A\vec{y}$ ✓

$k\vec{x} = \begin{bmatrix} kx_1 \\ kx_2 \\ kx_3 \end{bmatrix}$ $A(k\vec{x}) = \begin{bmatrix} kx_2 \\ kx_1 + kx_3 \\ kx_2 + 2kx_3 \end{bmatrix}$ $kA\vec{x} = k\begin{bmatrix} x_2 \\ x_1 + x_3 \\ x_2 + 2x_3 \end{bmatrix} = \begin{bmatrix} kx_2 \\ kx_1 + kx_3 \\ kx_2 + 2kx_3 \end{bmatrix} = A(k\vec{x})$ ✓

$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ $A(0) = \begin{bmatrix} 0 \\ 0+0 \\ 0+2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ ✓

yes, it satisfies the definition of a linear transformation

14. Answer the following questions as fully as possible, and justify your answer. (5 points each)

a. Explain why an nxn matrix can be both one-to-one and onto, but an mxn matrix where m≠n cannot be.

an nxn matrix w/ n pivots will have a pivot in every column (one-to-one) and in every column (onto).
however, if m≠n then one of the dimensions will be smaller than the other, meaning pivots in all of one is possible, but it will fall short other way.

b. Use general matrix properties to show that $(ABC)^T = C^T B^T A^T$.

$(ABC)^T = [(A)(BC)]^T = (BC)^T A^T = (C^T B^T) A^T$

$= C^T B^T A^T$

c. If A is a 5x3 matrix with three pivot positions, does the equation $A\vec{x} = \vec{0}$ have a solution? If so, is it trivial or non-trivial

it does have a solution since all homogenous systems are consistent. and w/ 3 pivots, it is also unique since A is one-to-one

d. Determine if the matrix $\begin{bmatrix} -5 & 1 & 4 \\ 0 & 0 & 0 \\ 1 & 4 & 9 \end{bmatrix}$ is invertible. Explain why or why not.

it is not since it has a row of 0's and this is missing a pivot.