1/30/2019

Review the first chapter in the book (online, this is Chapter 1, in the book it is all the material prior to limits). All this material will come up at some point in the course. Keep the review material in mind for when you run into it while we do the calculus. Do some practice problems in MyOpenMath or on your own to find out how much of a refresher you need.

While typically, multivariable functions are done last, I've incorporated them into the rest of the course. We aren't doing anything more than the basics, so we'll introduce it where it applies as needed.

We will start with multivariable functions.

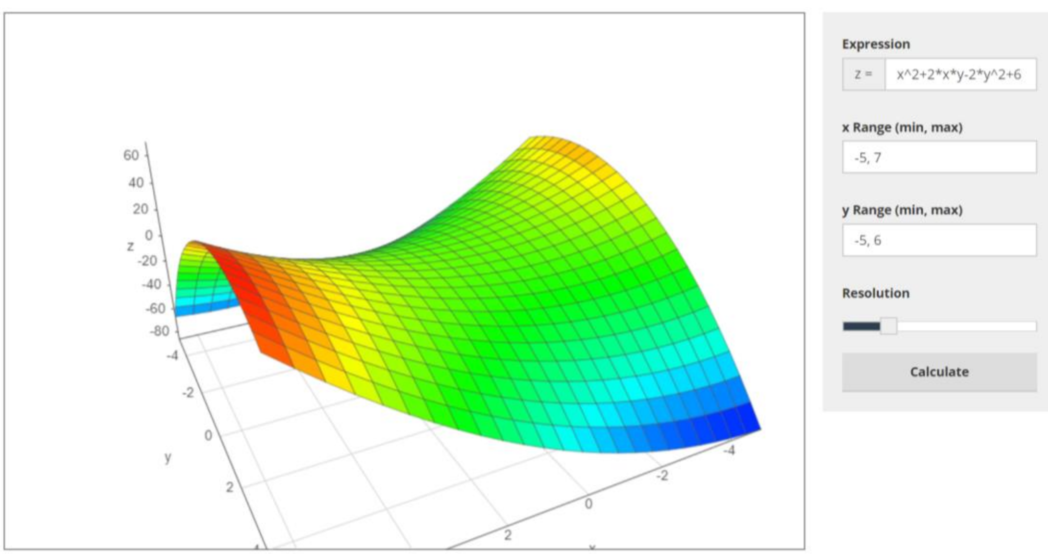
A function of two variables like $f(x,y)$ is a function with two independent variables x and y, and one dependent variable $f(x,y)=z$. So, this kind of function produces a graph in three variables and represents a surface.

We can have functions of more than two variables like $w=f(x,y,z)$, which we can work with and calculate values for, but which we can't visualize easily because now it is in four dimensions.

Suppose we have the function $f(x,y)=x^2+2xy-2y^2+6$. We can find values for $f(1,2)$, $and\ f(-2,3)$.

$$f(1,2)=1^2+2(1)(2)-2(2)^2+6=1+4-8+6=3$$
$$f(-2,3)=(-2)^2+2(-2)(3)-2(3)^2+6=4-12-18+6=-20$$

You can visualize what this surface looks like if you go to https://www.geogebra.org/3d?lang=en, enter the equation as $z=x^2+2xy-2y^2+6$, and drag your cursor around to see the graph from different angles. You'll see, the graph looks a bit like a saddle curve. Or go to

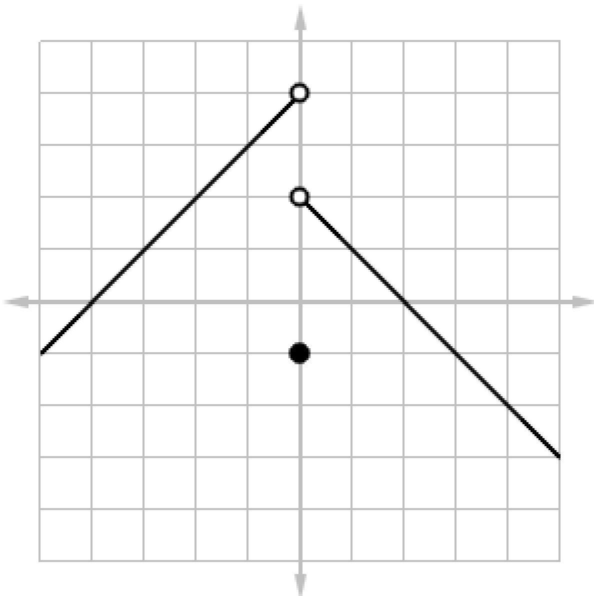
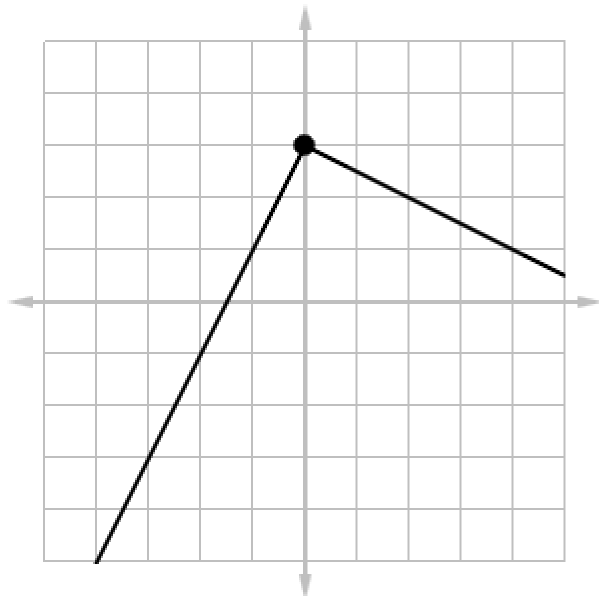


https://academo.org/demos/3d-surface-plotter/

You can find more on this topic here: https://www.khanacademy.org/math/multivariable-calculus/thinking-about-multivariable-function/ways-to-represent-multivariable-functions/a/multivariable-functions

Limits.

A limit is a way of saying what the value of the function (y) is approaching when x gets closer and closer to a given value. If the graph is continuous, then it's just the value of the function. If the graph is not continuous then the limit might exist or it might not. Let's consider a piecewise graph, since that's generally where we can run into non-continuous functions.

This is the graph of $f(x) = \begin{cases} x+4, x<0 \\ -1, \quad x=0 \\ 2-x, x>0 \end{cases}$ and $g(x) = \begin{cases} 2x+3, x<0 \\ 3-\frac{1}{2}x, \ x \geq 0 \end{cases}$.

If we want to find the limit of $g(x)$ at 0, we write this $\lim_{x\to 0} g(x)$ we see that as I get closer to $x = 0$ on either side of zero, the y-value gets closer and closer to 3. So $\lim_{x\to 0} g(x) = 3$ and separately we can say that $\lim_{x\to 0^-} g(x)$ (the limit from the left) and $\lim_{x\to 0^+} g(x)$ (the limit from the right) are the same and both are 3. Moreover, since $g(0) = 3$ also, the graph is continuous.

By contrast, if we look at $f(x)$, we see that if we get closer to 0 from the left, we get closer to 4. But if we come along the function from the right toward 0, we end up near 2. That means that the limit at 0 does not exist because the two sides are different values. And, the function value $f(0) = -1$, is also different, so the graph is not continuous. $\lim_{x\to 0} f(x) = DNE$.

You can do this algebraically by testing each piece at the endpoint, but it's easier once we have the graph already.

Some functions can be evaluated algebraically if they reduce, and sometimes we have to do them numerically because they don't simplify.

If I have function like $f(x) = \frac{x^2-4x-5}{x-5}$ and I want to know what the function is doing near $x = 5$, I can simplify by factoring. $f(x) = \frac{(x-5)(x+1)}{x-5} = x + 1$, and then plug in the value of $x = 5$ now that we aren't dividing by 0 anymore. Technically, this is a hole in the graph, but nearby, it will still behave like $x + 1$.

If we evaluate the function at values that get closer and closer to 5, we can produce the following table:

4	4.9	4.99	4.999	4.9999	5	5.0001	5.001	5.01	5.1	6	x
5	5.9	5.99	5.999	5.9999		6.0001	6.001	6.01	6.1	7	$f(x)$

What does it look like the value of the function is getting closer to? It looks like 6, which is the same value we got algebraically above, but this numerical approach can be used when simplifying won't.

The average rate of change

The average rate of change is essentially the slope between two points on a curve.

Consider $f(x) = x^2 - 3x + 8$. What is the average rate of change between $x = 2$ and $x = 5$.

First, we find the y-values for each point.

$$f(2) = 2^2 - 3(2) + 8 = 4 - 6 + 8 = 6$$
$$f(5) = 5^2 - 3(5) + 8 = 25 - 15 + 8 = 18$$

Then the average rate of change is $\frac{f(b)-f(a)}{b-a} = \frac{18-6}{5-2} = \frac{12}{3} = 4$.

Another way to describe the average rate of change is the slope of the secant line (a secant line connects two points on a graph).

The difference quotient is related to the average rate of change, but for an arbitrary point, and another point h units away.

$$\frac{f(x+h) - f(x)}{h}$$

For our function, this is
$$\frac{(x+h)^2-3(x+h)+8-(x^2-3x+8)}{h} = \frac{x^2+2xh+h^2-3x-3h+8-x^2+3x-8}{h} = \frac{2xh+h^2-3h}{h} = \frac{h(2x+h-3)}{h} = 2x + h - 3$$

Note: all the terms of the original function should cancel out, and then an h should factor out of every remaining term. If not, then there is an error in your math.

Related to the difference quotient is the limit definition of the derivative. As the two points gets closer and closer together, the slope of the line approaches a limit (for most of the functions we will consider, this limit exists). Closer together means the distance between the points, h, gets closer to zero, so the definition is:

$$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

For our function then

$$\lim_{h \to 0} 2x + h - 3 = 2x - 3$$

If I specify the value of x then I get the slope of the tangent line. (The tangent line is a line that is perpendicular to another line perpendicular to the graph at a particular point.)

If we don't evaluate it at a value, then the function we get is called the derivative of the function or $f'(x)$. You will also see the notation $\frac{df}{dx}$.

Limits: https://www.khanacademy.org/math/ap-calculus-ab/ab-limits-new/ab-1-2/v/introduction-to-limits-hd
Some more resources on average rate of change: http://home.windstream.net/okrebs/page201.html
Difference quotient: https://www.youtube.com/watch?v=1O5NEI8UuHM
Definition of the derivative : https://www.math24.net/definition-derivative/