MTH 261 Homework #6 Key

1a. $\int_0^1 \int_0^2 (x+y)\,dy\,dx = \int_0^1 xy + \frac{1}{2}y^2 \Big|_0^2 dx = \int_0^1 2x + 2\,dx = \left[x^2+2x\right|_0^1 = 3$

b. $\int_0^2 \int_{3y^2-6y}^{2y-y^2} 3y\,dx\,dy = \int_0^2 3xy \Big|_{3y^2-6y}^{2y-y^2} dy = \int_0^2 3y\left[2y-y^2-(3y^2-6y)\right]dy =$

$\int_0^2 3y(2y-y^2-3y^2+6y)\,dy = \int_0^2 3y(8y-4y^2)\,dy = \int_0^2 24y^2-12y^3\,dy =$

$8y^3-3y^4\Big|_0^2 = 64-48 = 16$

c. $\int_1^4 \int_1^{\sqrt{x}} 2ye^{-x}\,dy\,dx = \int_1^4 y^2e^{-x}\Big|_1^{\sqrt{x}} dx = \int_1^4 (x-1)e^{-x}\,dx$

$u=(x-1) \quad dv = e^{-x}dx \qquad -(x-1)e^{-x}\Big|_1^4 + \int_1^4 e^{-x}dx = -(x-1)e^{-x} - e^{-x}\Big|_1^4 =$

$du = dx \qquad v = -e^{-x} \qquad\qquad -(3)e^{-4} - e^{-4} + 0 + e^{-1} =$

$\frac{1}{e} - \frac{1}{e^4} - \frac{3}{e^4} = \frac{1}{e} - \frac{4}{e^4}$

d. $\int_0^4 \int_0^{x/2} dy\,dx + \int_4^6 \int_0^{6-x} dy\,dx = \int_0^4 \frac{x}{2}\,dx + \int_4^6 6-x\,dx =$

$\frac{1}{4}x^2\Big|_0^4 + (6x - \frac{1}{2}x^2\Big|_4^6 = 4 + 36 - 18 - 24 + 8 = 12 + 18 - 24 = 6$

2.
a. $\int_0^1 \int_0^2 cx(1+y)\,dy\,dx = \int_0^1 cx\,dx \int_0^2 1+y\,dy = \frac{c}{2}x^2\Big|_0^1 \; y + \frac{1}{2}y^2\Big|_0^2 =$

$\frac{c}{2}\underbrace{(2+2)}_{(4)} = 2c \Rightarrow c = \frac{1}{2}$

$P(x \geq \frac{1}{2}) =$

$\int_{1/2}^1 \frac{1}{2}x\,dx \int_0^2 1+y\,dy = \frac{1}{4}x^2\Big|_{1/2}^1 (4) = \cancel{\frac{1}{4}}(1-\frac{1}{4})\cancel{(4)} = \frac{3}{4}$

$(P\, x \geq \frac{1}{2}, y \leq \frac{1}{2})$

$\int_{1/2}^1 \frac{1}{2}x\,dx \int_0^{1/2} 1+y\,dy = \frac{3}{4}(y + \frac{1}{2}y^2)\Big|_0^{1/2} = \frac{3}{4}\left(\frac{1}{2}+\frac{1}{8}\right) = \frac{15}{32}$

$P(x+y \leq 1) \qquad y \leq 1-x$

$\int_0^1 \int_0^{1-x} \frac{1}{2}x(1+y)\,dy\,dx = \int_0^1 \frac{1}{2}x(y^2+y\Big|_0^{1-x} dx = \int_0^1 \frac{1}{2}x\left[(1-x)^2+(1-x)\right]dx = \int_0^1 \frac{1}{2}x\left[1-2x+x^2+1-x\right]dx$

2a cont'd

$$\int_0^1 \tfrac{1}{2}x(x^2-3x+2)\,dx = \tfrac{1}{2}\int_0^1 x^3-3x^2+2x\,dx = \tfrac{1}{2}\left[\tfrac{1}{4}x^4 - x^3 + x^2\right]_0^1 =$$

$$\tfrac{1}{2}\left[\tfrac{1}{4}-1+1\right] = \tfrac{1}{8}$$

b. $$\int_0^\infty\int_0^\infty c e^{-(\frac{1}{2}x+\frac{1}{5}y)}\,dy\,dx = c\int_0^\infty e^{-1/2x}\,dx \int_0^\infty e^{-1/5y}\,dy = c(-2e^{-1/2x}\Big|_0^\infty(-5e^{-1/5y}\Big|_0^\infty$$

$$= c(-2)(-5)(0+1)(0+1) = 10c = 1 \Rightarrow c = \tfrac{1}{10}$$

$P(y \geq 1)$

$$\int_0^\infty \tfrac{1}{10} e^{-1/2x}_{dx}\int_1^\infty e^{-1/5y}\,dy = \cancel{\tfrac{1}{10}}\cancel{(2)}\cancel{5}(e^{-1/5}) = e^{-1/5} \approx 0.8187$$

$P(X \leq 2, Y \leq 4)$

$$\int_0^2 \tfrac{1}{10}e^{-1/2x}\,dx\int_0^4 e^{-1/5y}\,dy = \cancel{\tfrac{1}{10}}\cancel{(2)}(-e^{-1}+1)\cancel{(5)}(-e^{-4/5}+1) = (1-\tfrac{1}{e})(1-\tfrac{1}{e^{4/5}})$$

$$\approx 0.3481$$

3. $f(x) = k e^{-x^2/2}$ $\quad\left(k = \frac{1}{\sqrt{2\pi}}\right)$

$$k\int_{-\infty}^\infty x e^{-x^2/2}\,dx = k\left(-e^{-x^2/2}\Big|_{-\infty}^\infty = k\left[\lim_{b\to\infty}\left(-\cancel{e^{-b^2/2}}^{\,0}+1\right) + \lim_{b\to\infty}\left(-1+\cancel{e^{-b^2/2}}^{\,0}\right)\right]\right.$$

$$k\left[\int_{-\infty}^0 xe^{-x^2/2}\,dx + \int_0^\infty xe^{-x^2/2}\,dx\right] \qquad = k(1-1) = 0$$

the mean is 0.