Rabbits and Foxes

Most species live in interaction with other species. For example, perhaps one species preys on another species, like foxes and rabbits. Below is a **system of rate of change equations** intended to predict future populations of rabbits and foxes over time, where R is the population (in hundreds or thousands, for example) of rabbits at any time t and F is the population of foxes at any time t (in years).

$$\frac{dR}{dt} = 3R - 1.4RF$$
$$\frac{dF}{dt} = -F + 0.8RF$$

1. (a) In earlier work with the rate of change equation $\frac{dP}{dt} = kP$ we assumed that there was only one species, that the resources were unlimited, and that the species reproduced continuously. Which, if any, of these assumptions is modified and how is this modification reflected in the above system of differential equations?

more than one species, for foxes, resources are not unlimited since they will decay if alone

(b) Interpret the meaning of *each* term in the rate of change equations (e.g., how do you interpret or make sense of the $-1.4RF$ term) and what are the implications of this term on the future predicted populations? Similarly for $3R$, $-F$, and $0.8RF$.

the change in rabbits is reduced by 1.4 rabbits per fox per time
the change in foxes increases by 0.8 foxes per rabbit per time

2. (a) Scientists studying a rabbit-fox population estimate that the current number of rabbits is 1 (scaled appropriately) and that the scaled number of foxes is 1. Use two steps of Euler's method with step size of 0.5 to get numerical estimates for the future number of rabbits and foxes as predicted by the differential equations.

t	R	F
0	1	1
0.5	1.8	.9
1.0	3.4	1.1

$\frac{dR}{dt} = 3(1) - 1.4(1)(1) = 1.6$ $\quad 1 + 1.6(.5) = 1.8$

$\frac{dF}{dt} = -(1) + .8(1)(1) = -.2$ $\quad 1 + -.2(.5) = 0.9$

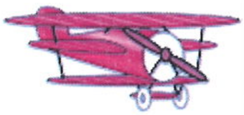
$\frac{dR}{dt} = 3(1.8) - 1.4(1.8)(.9) = 3.132$ $\quad 1.8 + 3.132(.5) = 3.366$

$\frac{dF}{dt} = -(.9) + .8(1.8)(.9) = .396$ $\quad .9 + .396(.5) = 1.098$

(b) What are some different two dimensional and three dimensional ways to graphically depict your (t, R, F) values?

a 3D graph with all t, R, F
2D R vs F, R vs t, F vs. t

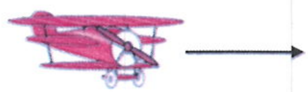
Three Dimensional Visualization



3. A crop duster plane with a two blade propeller is rolling down a runway. On the end of one of the propeller blades, which are rotating clockwise at a slow constant speed, is a noticeable red paint mark. Imagine that for the first several rotations of the propeller blades the red mark leaves a "trace" in the air as the plane makes its way down the runway.

(a) Simulate this scenario over time with a pipe cleaner. On appropriate combinations of the x, y, and t axes, sketch what Angler, Sider, Fronter, and Topper would ideally see assuming that they could always see the red mark. What view do you think is best and why?

Topper is directly above the runway in a hot air balloon moving with the airplane



Fronter is on a truck moving at the same speed as the airplane

Angler behind and off to the side of the airplane moving with the airplane

Sider is on the runway moving with the airplane from the side

Sketch your ideas for each of the following:

(b) What if there was another paint mark on other end of the propeller, what, **ideally**, do the four observers see then? How does the trace of this mark relate to the previous trace?

Fronter & Angler look same w/ 2 dots, Topper & Sider see them slide past each other on the line

(c) What if there was a paint mark on the center of the propeller blade mechanism. What do the observers ideally see then?

a dot turning w/ no movement

(d) How ideally would each observer see all of the above paint marks simultaneously?

plot out paintmarks over time, looks like a slinky

4. (a) For the system of differential equations from problem 1,

$$\frac{dR}{dt} = 3R - 1.4RF$$
$$\frac{dF}{dt} = -F + 0.8RF$$

consider the perspectives of Angler, Sider, Fronter, and Topper. What are the coordinate axes that correspond to each?

fronter is RF   Topper/Sider F,t or R,t

angler needs all three or RF at an angle

(b) Use the GeoGebra applet https://ggbm.at/U3U6MsyA to generate predictions for the future number of rabbits and foxes if at time 0 we initially have 1 rabbit and 3 foxes (scaled appropriately). Generate and reproduce below the perspectives of Angler, Sider, Fronter, and Topper from the crop duster problem.

slinky

(c) Use the same GeoGebra applet from problem 4b to experiment with different initial conditions and interpret the nature of the numerical solutions in the context of Rabbits and Foxes.

"Stable" orbit, decays to an equilibrium

(d) Determine an initial rabbit and fox population at time 0 such that the 3D graph of the solution (Angler's view) is a shift of the 3D graph in problem 4b along the t-axis. What connections does this problem have to do with your study of autonomous first order differential equations?

any value one cycle later in time
answers will vary

5. (a) Suppose the current number of rabbits is 3 and the number of foxes is 0. Without using any technology and without making any calculations, what does the system of rate of change equations (same one as problem 4a) predict for the future number of rabbits and foxes? Explain your reasoning.

rabbits start w/ exponential growth. $\frac{dR}{dt} = 3R$

foxes stay dead $\frac{dF}{dt} = 0$

(b) Use the same GeoGebra applet from problem 4b to generate the 3D plot and all three different views or projections of the 3D plot. Show each graph and explain how each illustrates your conclusion in problem 5a.

answers will vary

6. (a) What would it mean for the rabbit-fox system to be in equilibrium? Are there any equilibrium solutions to this system of rate of change equations? If so, determine all equilibrium solutions and generate the 3D and other views for each equilibrium solution.

population of rabbits & foxes stay constant over time

(0,0) is equilibrium $\left(\frac{5}{4}, \frac{15}{7}\right)$ stable

(b) For single differential equations, we classified equilibrium solutions as attractors, repellers, and nodes. For each of the equilibrium solutions in the previous problem, create your own terms to classify the equilibrium solutions in 6a and briefly explain your reasons behind your choice of terms.

answers will vary

(0,0)
extinction

$\left(\frac{5}{4}, \frac{15}{7}\right)$
love-hate

7. A group of scientists wants to graphically display the predictions for many different non-negative initial conditions (this includes 0 values for R and F, but not negative values) to the rabbit-fox system of differential equations and they want to do so using only one set of axes. What one single set of axes would you recommend that they use ($R-F-t$ axes, $t-R$ axes, $t-F$ axes, or $R-F$ axes)? Explain.

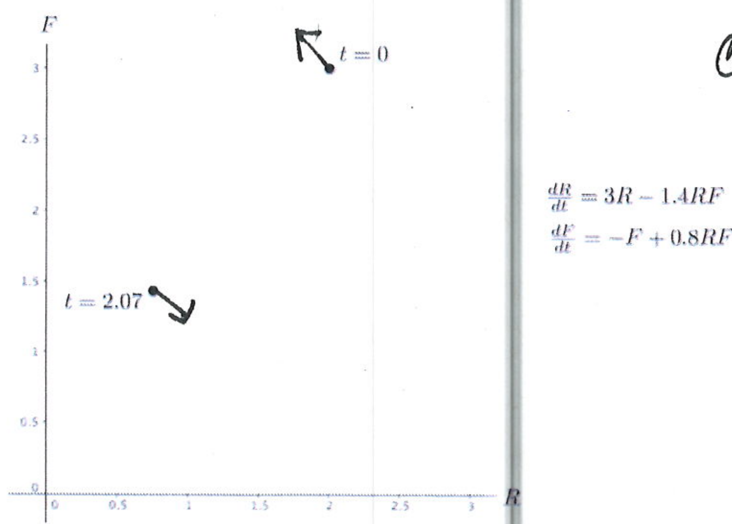
answers will vary

RF, w/ time as an arrow

8. One view of solutions for studying solutions to systems of autonomous differential equations is the $x - y$ plane, called the **phase plane**. The phase plane, which is Fronter's view from the crop duster problem, is the analog to the phase line for a single autonomous differential equation.

   (a) Consider the rabbit-fox system of differential equations and a solution graph, as viewed in the phase plane (that is, the R-F plane), and the two points in the table below. These two points are on the same solution curve. Recall that the solutions we've seen in the past are closed curves, but notice that the solution could be moving clockwise / counterclockwise. Fill in the following table and decide which way the solution should be moving, and explain your reasoning.

t	R	F	dR/dt	dF/dt	dF/dR
0	2	3	-2.4	1.8	-.75
2.07	0.756	1.431	.753	-.566	-1.33



Counter clockwise

   (b) On the same set of axes from problem 8a plot additional vectors at the following points and state what is unique about these vectors.

R	F	dR/dt	dF/dt	dF/dR
1.25	0	3.75	0	0
1.25	1	2	0	0
1.25	2	.25	0	0

slopes are horizontal

Vector Fields

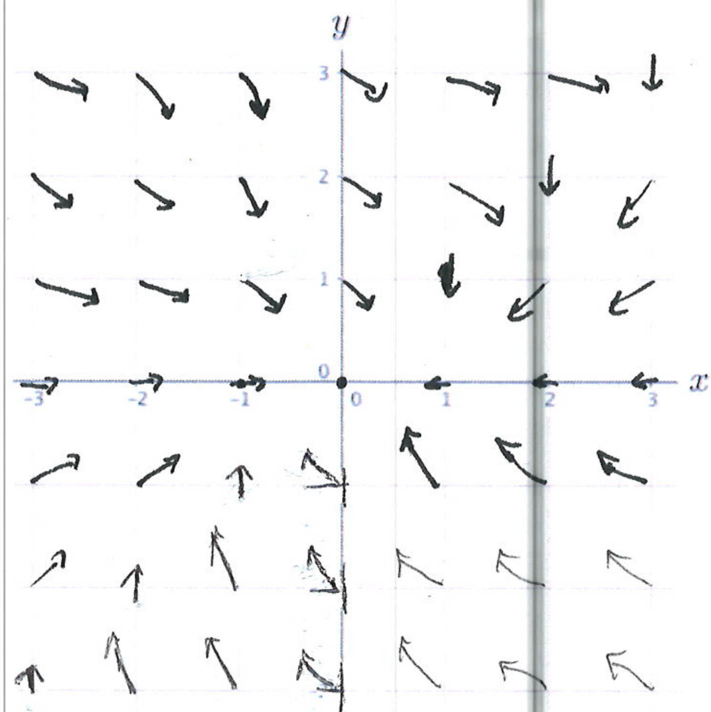
Slope fields are a convenient way to visualize solutions to a single differential equation. For systems of autonomous differential equations the equivalent representation is a **vector field**. Similar to a slope field, a vector field shows a selection of vectors with the correct slope but with a normalized length. In the previous problem you plotted a few such vectors but typically more vectors are needed to be able to visualize the solution in the phase plane.

9. On a grid where x and y both range from -3 to 3, plot by hand a vector field for the system of differential equations

$$\frac{dx}{dt} = y - x$$
$$\frac{dy}{dt} = -y$$

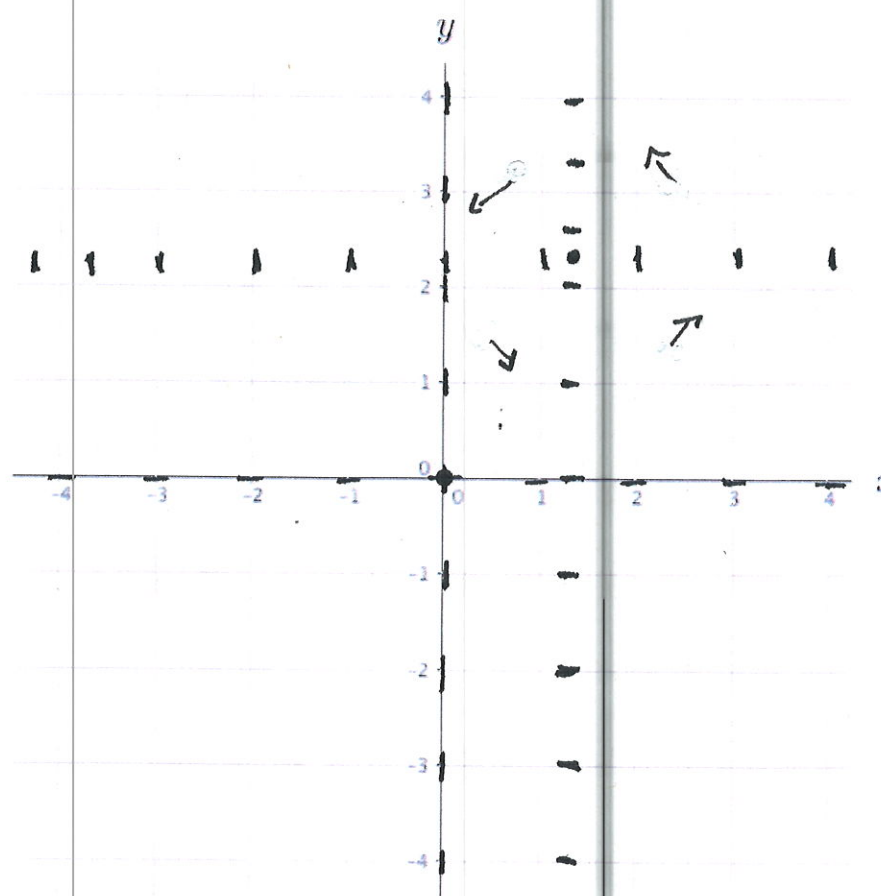
$$\frac{dy}{dx} = \frac{-y}{y-x}$$

and sketch in several solution graphs in the phase plane.



10. (a) You may have noticed in problem 9 that along $x = 0$ all the vectors have the same slope. Similarly for vectors along the $y = x$. Any line or curve along which vectors all have the same slope is called an **isocline**. An isocline where $dx/dt = 0$ is called an **x-nullcline** because there is the horizontal component to the vector is zero and hence the vector points straight up or down. An isocline where $dy/dt = 0$ is called a **y-nullcline** because the vertical component of the vector is zero and hence the vector points left or right. On a grid from -4 to 4 for both axes, plot all nullclines for the following system:

$$\frac{dx}{dt} = 3x - 1.4xy \qquad x(3-1.4y) \qquad y=\frac{3}{1.4}=\frac{15}{7}$$
$$\frac{dy}{dt} = -y + 0.8xy \qquad y(-1+.8x) \qquad x=\frac{1}{.8}=\frac{5}{4}$$



(b) How do these nullclines point to the cyclic nature of the Rabbit-Fox system?

answers will vary but discuss vectors around equilibrium

11. A certain system of differential equations for the variables R and S describes the interaction of rabbits and sheep grazing in the same field. On the phase plane below, dashed lines show the R and S nullclines along with their corresponding vectors.

    (a) Identify the R nullclines and explain how you know.

    (b) Identify the S nullclines and explain how you know.

    (c) Identify all equilibrium points.

    (d) Notice that the nullclines carve out 4 different regions of the first quadrant of the RS plane. In each of these 4 regions, add a prototypical-vector that represents the vectors in that region. That is, if you think the both R and S are increasing in a certain region then, draw a vector pointing up and to the right for that region.

    (e) What does this system seem to predict will happen to the rabbits and sheep in this field?