MTH 291, Final Exam, Spring 2018 Name KEY

Instructions: Show all work. Give exact answers unless specifically asked to round. All complex numbers should be stated in standard form, and all complex fractions should be simplified. If you do not show work, problems will be graded as "all or nothing" for the answer only; partial credit will not be possible and any credit awarded for the work will not be available.

1. Identify the Ansatz to find the particular solution for the differential equation $y'' + 4y' - 12y = f(x)$. (8 points each)

   a. $f(x) = x^2 - 3$

   $Ax^2 + Bx + C$

   $r^2 + 4r - 12 = 0$

   $(r+6)(r-2) = 0$

   $r = -6, 2$

   b. $f(x) = 2e^{2x}$

   Axe^{2x}

   c. $f(x) = 9xe^{-x}\sin 2x$

   $Axe^{-x}\sin 2x + Bxe^{-x}\cos 2x + Ce^{-x}\sin 2x + De^{-x}\cos 2x$

2. Consider the equation $\frac{d^2x}{dt^2} + xe^{0.1x} = 0$. Linearize the equation and perform a stability analysis. (14 points)

   $\frac{dx}{dt} = y$

   $\frac{dy}{dt} = -x$

   $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

   $-xe^{.1x} = -x\sum_{n=0}^{\infty}\frac{(.1x)^n}{n!} = -\sum_{n=0}^{\infty}\frac{x^{n+1}(.1)^n}{n!} = -x$

   $\begin{bmatrix} -\lambda & 1 \\ -1 & -\lambda \end{bmatrix} = \lambda^2 + 1 = 0$ $\lambda = \pm i$

   Since real part is zero, this is a stable orbit

   we would need another term to see more trends

3. Discuss and illustrate with examples how to solve differential equation of the form $\frac{dy}{dx} = f(x)$ and $\frac{d^2x}{dt^2} = f(x)$. (15 points)

for $\frac{dy}{dx} = f(x)$ since $f(x)$ is a function of x alone integrate to find y. use one initial condition to determine the constant.

for $\frac{d^2x}{dt^2} = f(x)$ apply separation of variables to first find $\frac{dx}{dt}$ and then again to find $x(t)$.

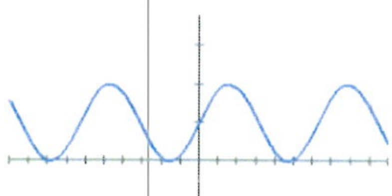
2 initial conditions will be needed to find all constants of integration.

4. Without solving, match the graph of a solution curve of $y'' + y = f(x)$ to one of the functions (you won't use them all). Do any of the solutions exhibit beats or resonance? (8 points)

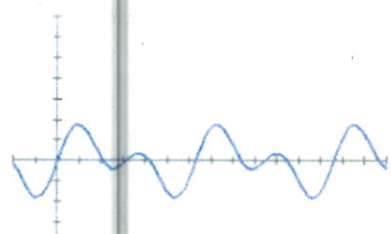
a. $f(x) = 1$ i
b. $f(x) = e^x$
c. $f(x) = e^x \sin x$
d. $f(x) = e^{-x}$ ii
e. $f(x) = \sin 2x$ iii
f. $f(x) = \sin x$ iv

$r^2 + 1 = 0$
$r = \pm i$ $\sin x, \cos x$

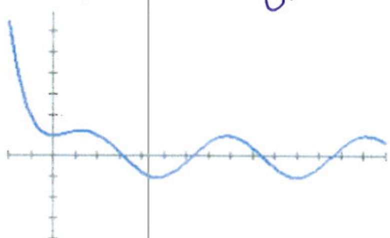
A



i. vertical shift

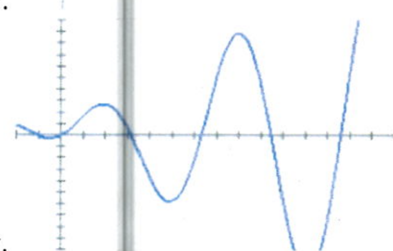


iii. E beats



ii.

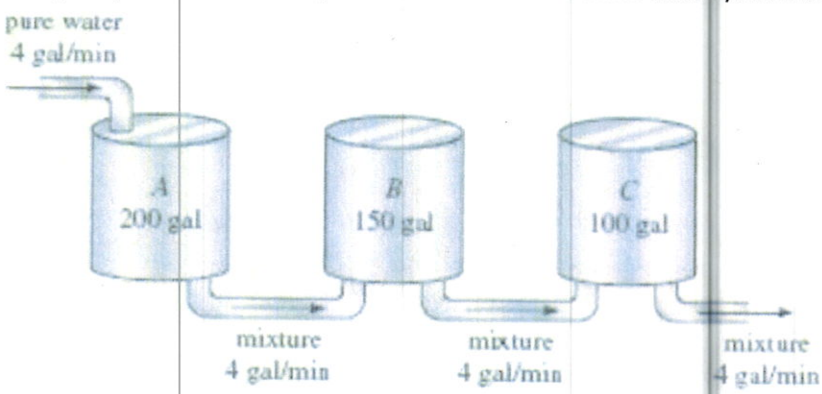
D
exponential decay of forcing reduces to normal sinusoidal



iv.

resonance
$\boxed{F}$

5. Set up a system of linear equations to model the brine tank system below. Do not solve. (15 points)



$$\frac{dA}{dt} = 0 - \frac{4A}{200}$$

$$\frac{dB}{dt} = \frac{4A}{200} - \frac{4B}{150}$$

$$\frac{dC}{dt} = \frac{4B}{150} - \frac{4C}{100}$$

6. The solution to $\frac{dy}{dx} = 6x^2 - 3x^2y, y(0) = 3$ is $y = 2 + e^{-x^3}$. Compute the value of $y(1)$ using $\Delta x = 0.2$ with Euler's Method. The find the value of the exact solution. Compare your estimate and calculate the error. (25 points)

n	x_n	y_n	f_n	y_{n+1}
0	0	3	$6(0)^2-3(0)^2(3)=0$	$0(.2)+3=3$
1	.2	3	$6(.2)^2-3(.2)^2(3)=-.12$	$-.12(.2)+3 = 2.976$
2	.4	2.976	$6(.4)^2-3(.4)^2(2.976)= -.46848$	$-.46848(.2)+2.976 = 2.8823$
3	.6	2.8823	$6(.6)^2-3(.6)^2(2.8823) = -.95288$	$-.95288(.2)+2.8823 = 2.6917$
4	.8	2.6917	$6(.8)^2-3(.8)^2(2.6917) = -1.328$	$-1.328(.2)+2.6917 = 2.426$
5	1.0	2.426		

$y(1) \approx 2.426$

vs.

$2 + e^{-1^3} = 2.3678\ldots$

$|y - y_{est}| \approx .058$

7. Solve the differential equation $\frac{dy}{d\theta} = \frac{e^y \sin\theta}{y \sec\theta}$ using separation of variables. (16 points)

$$\int y e^{-y} dy = \int \sin\theta \cos\theta \, d\theta$$

$u = y \quad dv = e^{-y}$
$du = dy \quad v = -e^{-y}$

$$-ye^{-y} + \int e^{-y} dy = \frac{1}{2}\sin^2\theta + C$$

$$-ye^{-y} - e^{-y} = \frac{1}{2}\sin^2\theta + C$$

8. Solve the differential equation $\frac{dp}{dt} = t^2 p - p + t^2 - 1$ using the method of integrating factors (reverse product rule). (16 points)

$$\frac{dp}{dt} = p(t^2-1) + 1(t^2-1) = (p+1)(t^2-1)$$

$$\frac{dp}{dt} - p(t^2-1) = t^2 - 1$$

$\mu = e^{-\int t^2 - 1 \, dt}$

$\mu = e^{-(\frac{1}{3}t^3 - t)}$

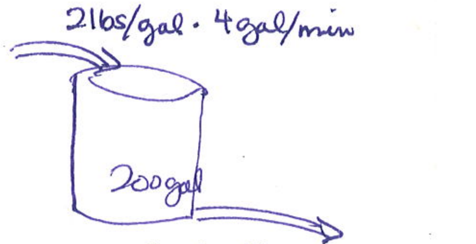
$$e^{-(\frac{1}{3}t^3 - t)} \frac{dp}{dt} - (t^2-1) e^{-(\frac{1}{3}t^2 - t)} p = e^{-(\frac{1}{3}t^2 - t)} (t^2-1)$$

$$\int \left(e^{-(\frac{1}{3}t^2 - t)} p\right)' = \int e^{-(\frac{1}{3}t^3 - t)} (t^2-1) \, dt$$

$$\cancel{e^{-(\frac{1}{3}t^2 - t)}} p = \left(\cancel{e^{-(\frac{1}{3}t^3 - t)}} + C\right) \cdot e^{\frac{1}{3}t^3 - t}$$

$$p = Ce^{\frac{1}{3}t^3 - t} + 1$$

9. A tank with 200 gallons of brine solution contains 40 lbs of salt. A concentration of 2 lb/gal is pumped in at a rate of 4 gal/min. The concentration leaving the tank is pumped out at a rate of 4 gal/min. How much salt is in the tank after 1 hour? How much salt is in the tank after a very long time? Sketch the graph of the solution. (16 points)



$A(0)=40$ $\quad \frac{A}{200}\cdot\frac{4gal}{min}$

$A(60)=?$

$$\frac{dA}{dt} = 2.4 - \frac{4A}{200}$$

$$\frac{dA}{dt} = 8 - \frac{A}{50} = -\frac{1}{50}(A-400)$$

$$\int \frac{dA}{A-400} = \int -\frac{1}{50}\,dt$$

$$\ln|A-400| = -\frac{1}{50}t + C$$

$$A - 400 = A_0 e^{-1/50 t}$$

$$A(t) = 400 + A_0 e^{-1/50 t}$$

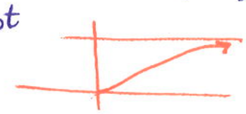
$$40 = 400 + A_0 e^{0}$$

$$A_0 = 360$$

$$A(t) = 400 - 360e^{-1/50 t}$$

long term $= 400$ lbs

$A(60) = 291$ lbs.



10. Consider the competition model $\begin{cases} \frac{dx}{dt} = 0.4x + 0.2x^2 - xy \\ \frac{dy}{dt} = 0.6y - 1.2y^2 - xy \end{cases}$, $x(0) = 13, y(0) = 22$. Sketch the nullclines for the system and use them to identify any equilibria. Using the information obtained from the nullclines, can you characterize the equilibria as stable, unstable or a saddle point? (20 points)

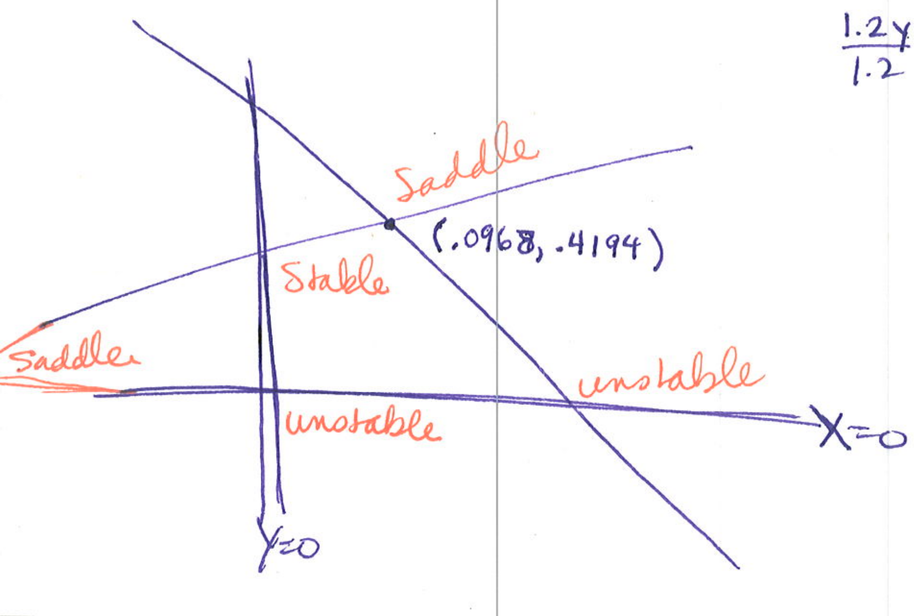
$$0 = .4x + .2x^2 - xy \qquad x=0$$

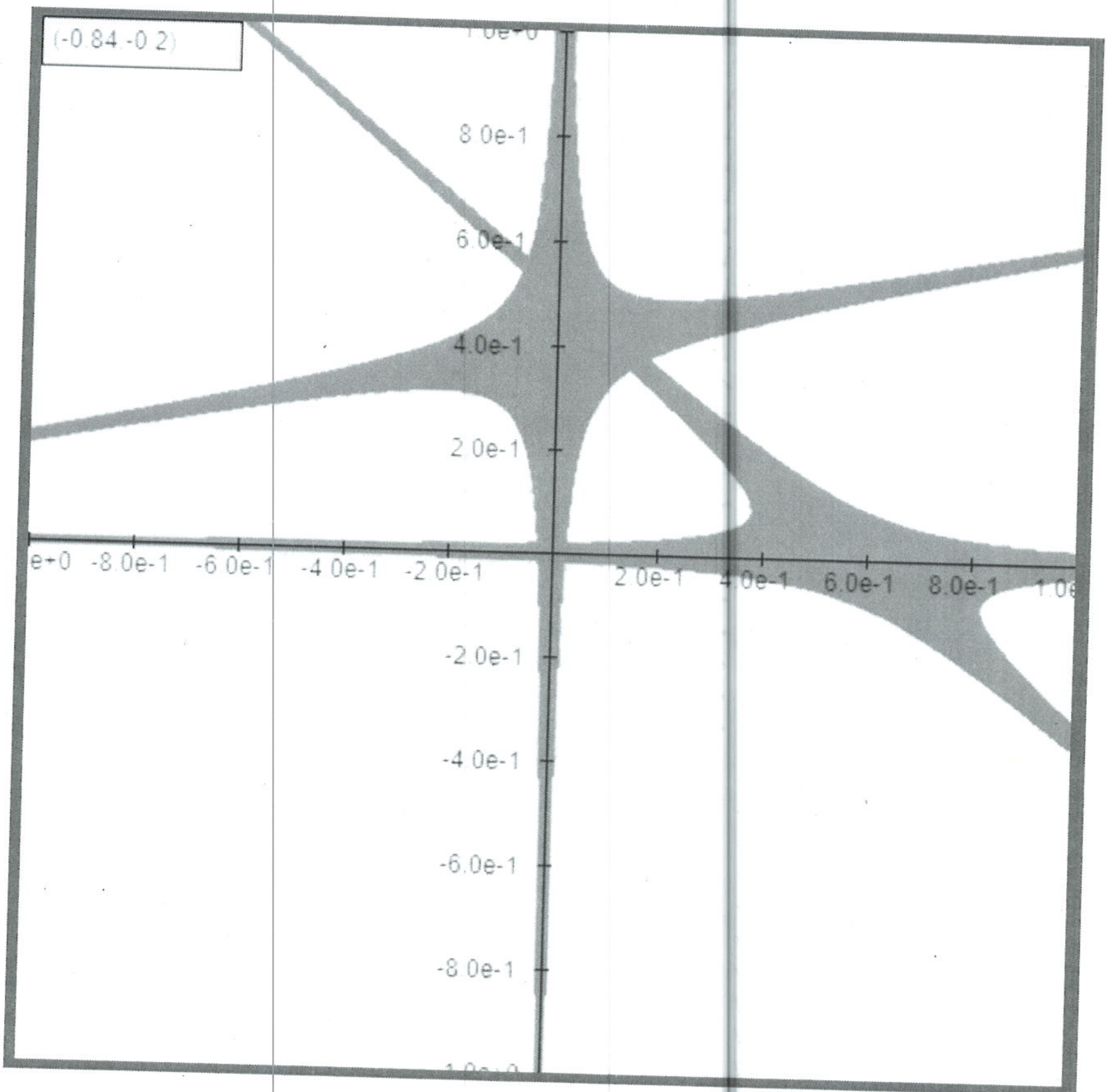
$$x(.4 + .2x - y) = 0 \qquad y = .4 + .2x$$

$$0 = .6y - 1.2y^2 - xy \qquad y=0$$

$$y(.6 - 1.2y - x) = 0 \qquad y = .5 - .8\bar{3}x$$

$$\frac{1.2y}{1.2} = \frac{.6 - x}{1.2}$$




(-0.84,-0.2)
8.0e-1
6.0e-1
4.0e-1
2.0e-1
-8.0e-1
-6.0e-1
-4.0e-1
-2.0e-1
2.0e-1
4.0e-1
6.0e-1
8.0e-1
-2.0e-1
-4.0e-1
-6.0e-1
-8.0e-1

11. Solve the systems of equations for the general solution below using eigenvalues. Be sure that your solutions are expressed only with real-valued functions. (20 points)

$$\vec{x}'(t) = \begin{pmatrix} -2 & 5 \\ -2 & 4 \end{pmatrix} \vec{x}$$

$(-2-\lambda)(4-\lambda) + 10 = 0$

$\lambda^2 - 2\lambda - 8 + 10 = 0$

$\lambda^2 - 2\lambda + 2 = 0$

$\lambda = \dfrac{2 \pm \sqrt{4-8}^{\,\leftarrow 2i}}{2} = 1 \pm i$

$\begin{bmatrix} -2-(1+i) & 5 \\ -2 & 4-(1+i) \end{bmatrix} = \begin{bmatrix} -3-i & 5 \\ -2 & 3-i \end{bmatrix}$

$-2x_1 + (3-i)x_2 = 0$

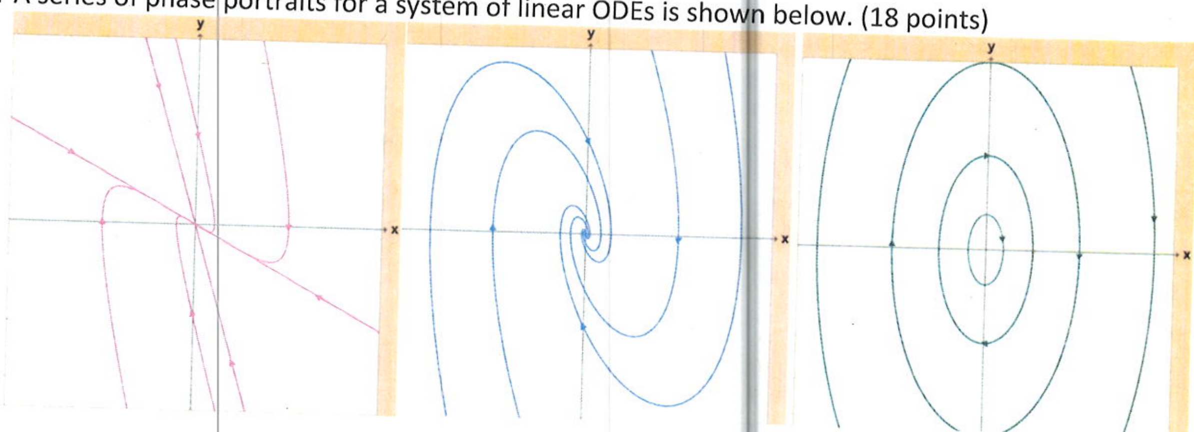
$x_1 = \dfrac{\cancel{-}(3-i)}{\cancel{-}2} x_2 \qquad \begin{bmatrix} 3-i \\ 2 \end{bmatrix}$

$\begin{bmatrix} 3-i \\ 2 \end{bmatrix} e^t (\cos t + i \sin t) =$

$e^t \begin{bmatrix} 3\cos t + 3i\sin t - i\cos t + \sin t \\ 2\cos t + 2i\sin t \end{bmatrix}$

$\vec{x}(t) = c_1 e^t \begin{bmatrix} 3\cos t + \sin t \\ 2\cos t \end{bmatrix} + c_2 e^t \begin{bmatrix} 3\sin t - \cos t \\ 2\sin t \end{bmatrix}$

12. A series of phase portraits for a system of linear ODEs is shown below. (18 points)



A B C

Each of the phase portraits is a possible solution for a system of springs. Describe the damping of each type of system. Explain your reasoning.

A = overdamped (possibly critical)

B = underdamped

C = undamped

13. Solve the second order differential equation $y'' + 6y' + 9y = 0$. (14 points)

$$r^2 + 6r + 9 = 0$$
$$(r+3)^2 = 0$$
$$r = -3$$

$$y = c_1 e^{-3t} + c_2 t e^{-3t}$$

14. Use the Existence and Uniqueness Theorem to determine where the differential equation $y' = \frac{2x}{\sqrt{1-y^2}}$ is guaranteed to have a unique solution. Sketch the graph of the region. (14 points)

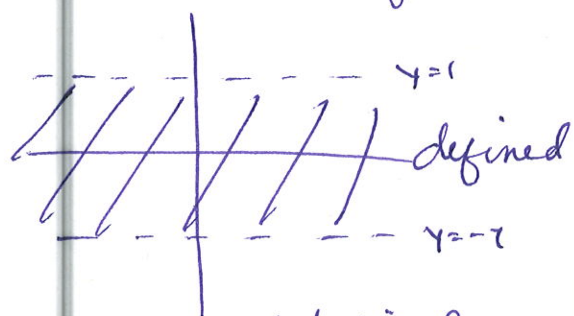
$f(x,y) = \frac{2x}{\sqrt{1-y^2}}$ $\qquad \sqrt{1-y^2} \neq 0$

$2x(1-y^2)^{-1/2}$ $\qquad 1-y^2 > 0$

$\qquad y^2 < 1$

$\frac{\partial f}{\partial y} = 2x \cdot \left(-\frac{1}{2}\right)(1-y^2)^{-3/2}(-2y)$ $\qquad -1 < y < 1$

no new pts.



15. Draw a phase line for the autonomous differential equation $y' = y^2(y^2-4)(y-4)$. Characterize each equilibrium as stable, unstable or semi-stable. (16 points)

$y^2(y-2)(y+2)(y-4)$

4 unstable

2 stable

0 semi-stable

-2 unstable