MTH 277 Homework #5 Key

1. See attached

2. a = iv b = ii c = v d = viii e = i f = vi g = vii h = iii

3. a. $f(x,y) = e^{xy}$

$\nabla f = \langle ye^{xy}, xe^{xy} \rangle$

$\nabla^2 f = y^2 e^{xy} + x^2 e^{xy}$

b. $f(x,y,z) = x\ln(y-2z)$

$\nabla f = \left\langle \ln(y-2z), \frac{x}{y-2z}, \frac{-2x}{y-2z} \right\rangle$

$\nabla^2 f = 0 + \frac{-x}{(y-2z)^2} + \frac{2x(-2)}{(y-2z)^2} = \frac{-x}{(y-2z)^2} - \frac{4x}{(y-2z)^2}$

c. $f(x,y) = x^2 - y^2 - 2x + 6y + 13$

$\nabla f = \langle 2x-2, -2y+6 \rangle$

$\nabla^2 f = 2 - 2 = 0$

d. $f(x,y,z) = \sqrt{x^2+y^2+z^2} = (x^2+y^2+z^2)^{1/2}$

$\nabla f = \langle (x^2+y^2+z^2)^{-1/2}(1/2)(2x), (x^2+y^2+z^2)^{-1/2}(1/2)(2y), (x^2+y^2+z^2)^{-1/2}(1/2)(2z) \rangle$

$= \left\langle \frac{x}{\sqrt{x^2+y^2+z^2}}, \frac{y}{\sqrt{x^2+y^2+z^2}}, \frac{z}{\sqrt{x^2+y^2+z^2}} \right\rangle$

$\nabla^2 f = (x^2+y^2+z^2)^{-1/2} + x\left(-\frac{1}{2}\right)(x^2+y^2+z^2)^{-3/2}(2x) + (x^2+y^2+z^2)^{-1/2} + (y)\left(-\frac{1}{2}\right)(x^2+y^2+z^2)^{-3/2}(2y)$

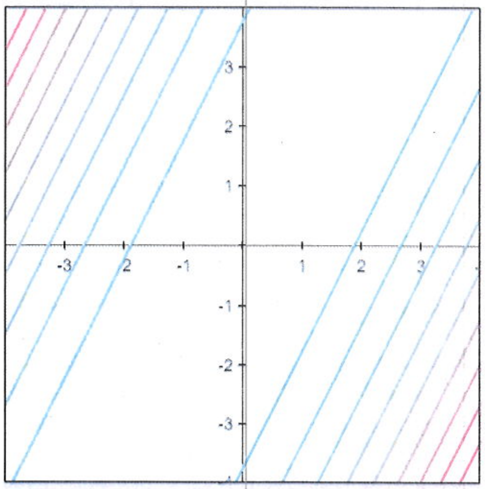
$+ (x^2+y^2+z^2)^{-1/2} + z\left(-\frac{1}{2}\right)(x^2+y^2+z^2)^{-3/2}(2z)$

$= \frac{3}{\sqrt{x^2+y^2+z^2}} - \frac{x^2+y^2+z^2}{(x^2+y^2+z^2)^{3/2}} = \frac{3}{\sqrt{x^2+y^2+z^2}} - \frac{1}{\sqrt{x^2+y^2+z^2}} = \boxed{\frac{2}{\sqrt{x^2+y^2+z^2}}}$

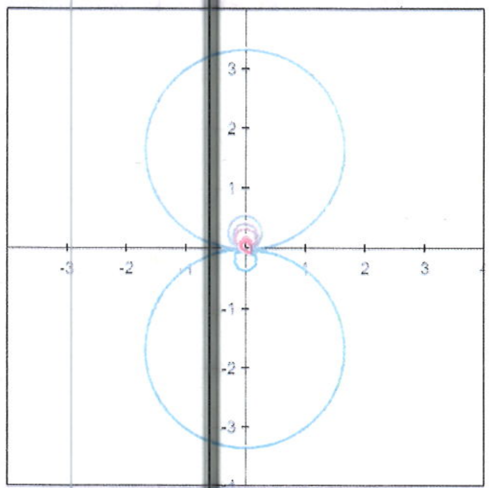
4a. $\vec{F}(x,y) = (xy-2)\hat{i} + (y^2-10)\hat{j}$

$\vec{\nabla}\cdot\vec{F} = y + 2y = 3y$

Homework #5 graphs

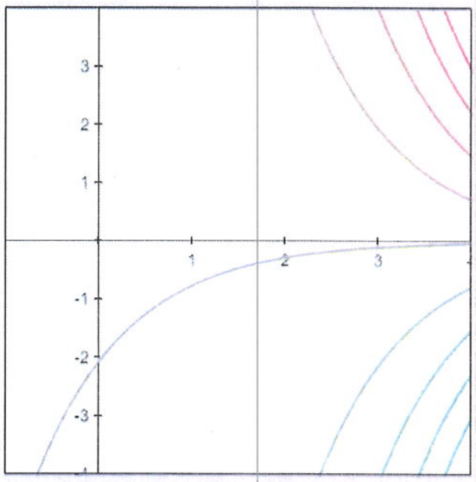


1a.

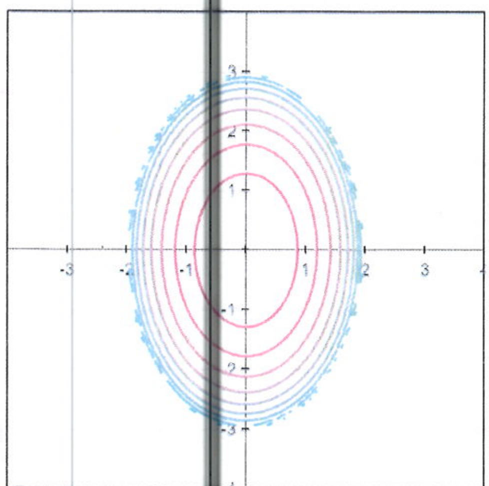


b.

1b graph is very sensitive to values (you may need to do values very close to 0 (on both sides of 0) to get a decent graph.



c.



d.

4a cont'd

$$\nabla \times F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xy-2 & y^2-10 & 0 \end{vmatrix} = (0-0)\hat{i} - (0-0)\hat{j} + (0-x)\hat{k} = -x\hat{k}$$

b. $F(x,y) = \tan(3x-4y)\hat{i} + \ln(1+x^2+2y^2)\hat{j}$

$$\vec{\nabla}\cdot F = 3\sec^2(3x-4y) + \frac{4y}{1+x^2+y^2}$$

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \tan(3x-4y) & \ln(1+x^2+2y^2) & 0 \end{vmatrix} = (0-0)\hat{i} + (0-0)\hat{j} + \left(\frac{2x}{1+x^2+2y^2} + 4\sec^2(3x-4y)\right)\hat{k}$$

c. $F(x,y,z) = xyz\,\hat{i} + (2x-3z)\hat{j} + (x^2+yz)\hat{k}$

$$\vec{\nabla}\cdot F = yz + 0 + y$$

$$\vec{\nabla}\times F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xyz & 2x-3z & x^2+yz \end{vmatrix} = (z+3)\hat{i} - (2x-xy)\hat{j} + (2-xz)\hat{k}$$

d. $F(x,y,z) = \arctan(yz)\hat{i} + e^{yz}\hat{j} + (z+1)^{1/3}\hat{k}$

$$\vec{\nabla}\cdot\vec{F} = 0 + ze^{yz} + \tfrac{1}{3}(z+1)^{-2/3}$$

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \arctan(yz) & e^{yz} & (z+1)^{1/3} \end{vmatrix} = (0-ye^{yz})\hat{i} - \left(0-\frac{y}{1+y^2z^2}\right)\hat{j} + \left(0-\frac{z}{1+y^2z^2}\right)\hat{k}$$

5a. $\vec{F}\times\vec{G}$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ xy & -2y & x^2\sin z \\ \ln x & 2e^z & -3y \end{vmatrix} = (6y^2 - 2x^2e^z\sin z)\hat{i} - (-3xy^2 - x^2\ln x\cdot\sin z)\hat{j} + (2xye^z + 2y\ln x)\hat{k}$$

$$\vec{\nabla}\times(F\times G) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 6y^2-2x^2e^z\sin z & 3xy^2+x^2\ln x\cdot\sin z & 2xye^z+2y\ln x \end{vmatrix} =$$

$$(2xe^z + 2\ln x - x^2\ln x\cos z)\hat{i} - \left(2ye^z + \frac{2y}{x} + 2x^2e^z\sin z + 2x^2e^z\cos z\right)\hat{j} + (3y^2 + 2x\ln x\cdot\sin z + x\sin z - 12y)\hat{k}$$

5b. see 5a for $F \times G$

$$\vec{\nabla}\cdot(F\times G) = (0 - 4xe^{z}\sin z) + (6xy + 0) + (2xye^{z} + 0)$$

$$= -4xe^{z}\sin z + 6xy + 2xye^{z}$$

5c. $\nabla f = \langle y, x, 2z \rangle$

$$\nabla\cdot(\nabla f) = \nabla^2 f = 0 + 0 + 2 = 2$$

d. $$\nabla \times F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xy & -2y & x^2\sin z \end{vmatrix} = (0-0)\hat{i} - (2x\sin z - 0)\hat{j} + (0 - x)\hat{k}$$

$$= \langle 0, -2x\sin z, -x \rangle$$

$$\vec{\nabla}\times(\vec{\nabla}\times F) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 0 & -2x\sin z & -x \end{vmatrix} = (0 + 2x\cos z)\hat{i} - (-1-0)\hat{j} + (-2\sin z - 0)\hat{k}$$

$$\langle 2x\cos z, 1, -2\sin z \rangle$$

e. $f\vec{F} = \langle x^2y^2 + xyz^2 - 6xy, -2xy^2 - 2yz^2 + 12y, x^3y\sin z + x^2z^2\sin z - 6x^2\sin z \rangle$

$$\vec{\nabla}\cdot(f\vec{F}) = 2xy^2 + yz^2 - 6y - 4xy - 2z^2 + 12 + x^3y\cos z + 2x^2z\sin z$$
$$+ x^2z^2\cos z - 6x^2\cos z$$

6a. $\nabla f = \langle 3\cos 3x \cos 4y, -4\sin 3x \sin 4y \rangle$

b. $\nabla f = \left\langle \arcsin yz, \dfrac{yz}{\sqrt{1-y^2z^2}}, \dfrac{xy}{\sqrt{1-y^2z^2}} \right\rangle$

c. $\nabla f = \left\langle -\dfrac{z}{x^2} - \dfrac{z}{y}, \dfrac{1}{z} + \dfrac{xz}{y^2}, -\dfrac{y}{z^2} + \dfrac{1}{x} - \dfrac{x}{y} \right\rangle$

7a. $F(x,y) = \dfrac{x}{\sqrt{x^2+y^2}}\hat{i} + \dfrac{y}{\sqrt{x^2+y^2}}\hat{j}$

$$\int \frac{x}{\sqrt{x^2+y^2}}\,dx = \int x(x^2+y^2)^{-1/2}\,dx \qquad u = (x^2+y^2) \quad du = 2x\,dx \qquad \int \tfrac{1}{2}u^{-1/2}\,du = u^{1/2} = \sqrt{x^2+y^2} + f(y)$$

$$\int \frac{y}{\sqrt{x^2+y^2}}\,dy = \sqrt{x^2+y^2} + g(x)$$

$$\varphi(x,y) = \sqrt{x^2+y^2} + K$$

7b. $\vec{F}(x,y) = 2xye^{x^2y}\hat{i} + x^2e^{x^2y}\hat{j}$

$\int 2xye^{x^2y}\,dx = e^{x^2y} + f(y)$ $\qquad$ $\int x^2e^{x^2y}\,dy = e^{x^2y} + g(x)$

$\varphi(x,y) = e^{x^2y} + K$

c. $F(x,y,z) = \sin y\,\hat{i} - x\cos y\,\hat{j} + \hat{k}$

$\int \sin y\,dx = x\sin y + f(y,z)$

$\int -x\cos y\,dy = -x\sin y + g(x,z)$

$\int 1\,dz = z + h(x,y)$

these do not match exactly
So no potential function exists.
field is not conservative
$\vec{\nabla}\times\vec{F} \neq \vec{0}$

d. $\vec{F}(x,y) = \frac{1}{x}\hat{i} - \frac{1}{y}\hat{j}$

$\int \frac{1}{x}\,dx = \ln x + f(y)$

$\int \frac{1}{y}\,dy = \ln y + g(x)$

$\varphi(x,y) = \ln x + \ln y + K$

or $\ln(xy) + K$

e. $\vec{F}(x,y) = 3x^2y^2\hat{i} + 3x^3y\,\hat{j}$

$\int 3x^2y^2\,dx = x^3y^2 + f(y)$

$\int 3x^3y\,dy = \frac{3}{2}x^3y^2 + g(x)$

these do not match exactly
So no potential function exists
field is not conservative
$\vec{\nabla}\times\vec{F} \neq \vec{0}$

f. $F(x,y,z) = y^2z^3\hat{i} + 2xyz^3\hat{j} + 3xy^2z^3\hat{k}$

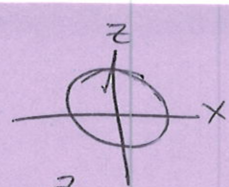
$\int y^2z^3\,dx = xy^2z^3 + f(y,z)$

$\int 2xyz^3\,dy = xy^2z^3 + g(x,z)$

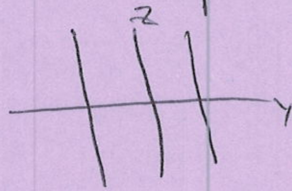
$\int 3xy^2z^3\,dz = \frac{3}{4}xy^2z^4 + h(x,y)$

these do not all match exactly
So no potential function exists
field is not conservative
$\vec{\nabla}\times\vec{F} \neq \vec{0}$

8a. $y = 0 \rightarrow x^2 + z^2 = 1$



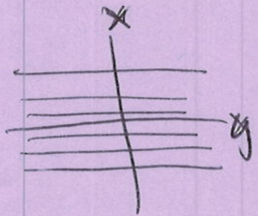
$x = 0 \qquad z^2 = 1$



$z =$ constant

$x^2 = 1 - c^2$

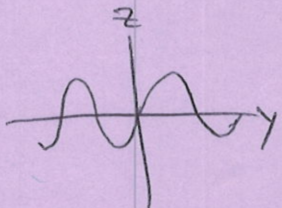
$x = \pm\sqrt{1-c^2}$



Surface is a cylinder wrapped around y-axis

b. $z = \sin y$

$x = 0$



$y = 0$

$z =$ constant

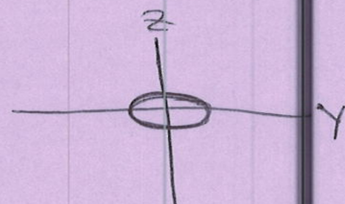
$y = \arcsin(c)$

Sheet of $\sin y$ stretching out in x direction (cylinder)

c. $x^2 + 4y^2 + 9z^2 = 1$

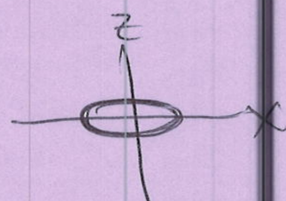
$x = 0 \rightarrow 4y^2 + 9z^2 = 1$

$= \dfrac{y^2}{(1/2)^2} + \dfrac{z^2}{(1/3)^2} = 1$



$y = 0 \rightarrow x^2 + 9z^2 = 1$

$\rightarrow \dfrac{x^2}{(1)^2} + \dfrac{z^2}{(1/3)^2} = 1$

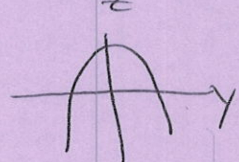


$z =$ constant

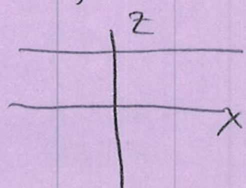
Ellipsoid

d. $z = 1 - y^2$

$x = 0$



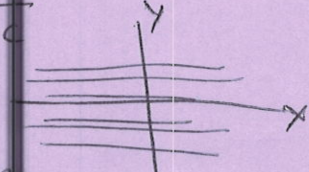
$y = 0 \rightarrow z = 1$



$z =$ constant

$y^2 = 1 - c$

$y = \pm\sqrt{1-c}$



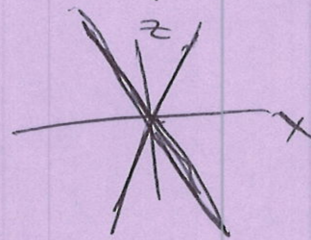
Sheet extending in x, parabolic cylinder

8e. $x^2 = y^2 + 4z^2$

$x = 0 \Rightarrow \quad 0 = y^2 + 4z^2$

$y = 0 \quad x^2 = 4z^2$

$x = \pm 2z$



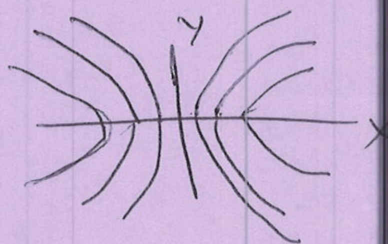
Cone wrapped around the x-axis

z = constant

$x^2 - y^2 = 4c^2$

$x^2 - 4c^2 = y^2$

$y = \pm\sqrt{x^2 - 4c^2}$

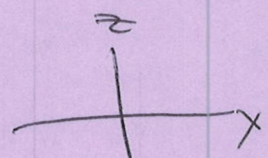


f. $-x^2 + y^2 - z^2 = 1$

$x = 0 \rightarrow y^2 - z^2 = 1$

hyperboloid of 2 sheets

$y = 0 \Rightarrow -x^2 - z^2 = 1$



no trace

z = constant

$y^2 - x^2 = 1 + c^2$

or $y = \pm\sqrt{1 + c^2 + x^2}$