MTH 277, Exam #3, Spring 2018 Name KEY

Instructions: Show all work. Give exact answers unless specifically asked to round. All complex numbers should be stated in standard form, and all complex fractions should be simplified. If you do not show work, problems will be graded as "all or nothing" for the answer only; partial credit will not be possible and any credit awarded for the work will not be available.

1. Find the unit tangent vector for $\vec{r}(t) = (t^3 - 4t)\hat{\imath} + (t^2 - 1)\hat{\jmath}$. (8 points)

$r'(t) = (3t^2-4)\hat{\imath} + (2t)\hat{\jmath}$

$\|r'(t)\| = \sqrt{(3t^2-4)^2 + (2t)^2} = \sqrt{9t^4 - 24t^2 + 16 + 4t^2}$

$= \sqrt{9t^4 - 20t^2 + 16}$

$\vec{T}(t) = \dfrac{(3t^2-4)\hat{\imath} + 2t\hat{\jmath}}{\sqrt{9t^4 - 20t^2 + 16}}$

2. Write an integral for the arc length of the curve $\vec{r}(t) = t\hat{\imath} + (4 - t^2)\hat{\jmath} + t^3\hat{k}$ on the interval $[0,2]$. Evaluate it numerically. (6 points)

$r'(t) = 1\hat{\imath} + (-2t)\hat{\jmath} + 3t^2\hat{k}$

$\|r'(t)\| = \sqrt{1 + 4t^2 + 9t^4}$

$s = \int_0^2 \sqrt{1 + 4t^2 + 9t^4}\, dt \approx \boxed{9.57}$

3. Find the curvature of the curve $\vec{r}(t) = 3t\hat{\imath} + e^t\hat{\jmath} + 2t^2\hat{k}$ at the point $t = 1$. What is the radius of curvature at the same point? (12 points)

$$r'(t) = 3\hat{i} + e^t\hat{j} + 4t\hat{k}$$
$$r''(t) = 0\hat{i} + e^t\hat{j} + 4\hat{k}$$

$$r' \times r'' = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & e^t & 4t \\ 0 & e^t & 4 \end{vmatrix} =$$

$$(4e^t - 4te^t)\hat{i} - (12 - 0)\hat{j} + (3e^t - 0)\hat{k}$$
$$4e^t(1-t)\hat{i} - 12\hat{j} + 3e^t\hat{k}$$

$$\|r' \times r''\| = \sqrt{16e^{2t}(1-t)^2 + 144 + 9e^{2t}}$$

$$\|r'(t)\| = \sqrt{9 + e^{2t} + 16t^2}$$

$$K = \frac{\|r' \times r''\|}{\|r'(t)\|^3} = \frac{\sqrt{16e^{2t}(1-t^2)\cancel{144}144 + 9e^{2t}}}{(9 + e^{2t} + 16t^2)^{3/2}}$$

$$K(1) = \frac{\sqrt{16(e^2)(0) + 144 + 9e^2}}{(9 + e^2 + 16)^{3/2}} = \frac{\sqrt{9e^2 + 144}}{(25 + e^2)^{3/2}}$$

$$\approx \frac{\sqrt{210.5}}{(32.4)^{3/2}} \approx \frac{14.5}{184.33} \approx .07866$$

$$K = .07866$$

$$\boxed{R \approx 12.71}$$

4. Find the directional derivative of the function $w = \ln \sqrt[3]{x^2 + y^2 + z^2}$ at the point (1,4,2) in the direction of $\vec{v} = \langle -2,6,2 \rangle$. In what direction is the directional derivative a maximum? (10 points)

$$w = \frac{1}{3}\ln(x^2 + y^2 + z^2)$$

$$\nabla w = \frac{1}{3} \cdot \frac{1}{x^2 + y^2 + z^2} \langle 2x, 2y, 2z \rangle$$

$$\nabla w(1,4,2) = \frac{1}{3} \cdot \frac{1}{1 + 16 + 4} \langle 2,8,4 \rangle = \frac{1}{63}\langle 2,8,4 \rangle$$

$$\hat{v} = \frac{\langle -2,6,2 \rangle}{\sqrt{4 + 36 + 4}} = \frac{\langle -2,6,2 \rangle}{\sqrt{44}} = \frac{\langle -2,6,2 \rangle}{2\sqrt{11}} = \frac{\langle -1,3,1 \rangle}{\sqrt{11}}$$

$$D_u w = \frac{1}{63}\langle 2,8,4 \rangle \cdot \frac{\langle -1,3,1 \rangle}{\sqrt{11}} = \frac{1}{63\sqrt{11}}(-2 + 24 + 4) = \frac{26}{63\sqrt{11}} \approx .1244$$

5. Find the equation of the tangent plane to the surface $x^2 + 2z^2 = y^2$ at $(1,3,-2)$. (8 points)

$F = x^2 + 2z^2 - y^2 = 0$

$\nabla F = \langle 2x, -2y, 4z \rangle$

$\nabla F(1,3,-2) = \langle 2, -6, -8 \rangle$

$2(x-1) - 6(y-3) - 8(z+2) = 0$

6. Find an equation of the tangent plane to the surface $\vec{r}(u,v) = 2u\cos v\,\hat{\imath} + 2u\sin v\,\hat{\jmath} + u^4\hat{k}$ at the point $u = 1, v = \frac{\pi}{6}$. (8 points)

$\langle 2\frac{\sqrt{3}}{2}, 2\cdot\frac{1}{2}, 1 \rangle$

$\langle \sqrt{3}, 1, 1 \rangle$

$r_u = 2\cos v\,\hat{\imath} + 2\sin v\,\hat{\jmath} + 4u^3\hat{k}$

$r_v = -2u\sin v\,\hat{\imath} + 2u\cos v\,\hat{\jmath} + 0\hat{k}$

$$r_u \times r_v = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2\cos v & 2\sin v & 4u^3 \\ -2u\sin v & 2u\cos v & 0 \end{vmatrix} =$$

$(0 - 8u^4\cos v)\hat{\imath} - (0 + 8u^4\sin v)\hat{\jmath} + (4u\cos^2 v + 4u\sin^2 v)\hat{k}$

$-8u^4\cos v\,\hat{\imath} - 8u^4\sin v\,\hat{\jmath} + 4u\,\hat{k}$

$r_u \times r_v(1, \pi/6) = \langle -8\cdot\frac{\sqrt{3}}{2}, -8\cdot\frac{1}{2}, 4 \rangle = \langle -4\sqrt{3}, -4, 4 \rangle$

$-4\sqrt{3}(x - \sqrt{3}) - 4(y-1) + 4(z-1) = 0$

7. Find the area of the surface given by $f(x,y) = 2 + \frac{2}{3}xy^{3/2}$ over the region $R: \{(x,y): 0 \le x \le 2, 0 \le y \le 2 - y\}$. Do not integrate. (8 points)

$$F = 2 + \frac{2}{3}xy^{3/2} - z$$

$$\nabla F = \langle \frac{2}{3}y^{3/2}, xy^{1/2}, -1 \rangle$$

$$\|\nabla F\| = \sqrt{\frac{4}{9}y^3 + x^2 y + 1}$$

$$\int_0^2 \int_0^{2-y} \sqrt{\frac{4}{9}y^3 + x^2y + 1}\, dx\, dy$$

8. Use the Fundamental Theorem of Line Integrals to evaluate $\int_C \cos x \sin y\, dx + \sin x \cos y\, dy$ on the line segment from $(0, -\pi)$ to $\left(\frac{3\pi}{2}, \frac{\pi}{2}\right)$. (10 points)

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & 0 \\ \cos x \sin y & \sin x \cos y & 0 \end{vmatrix} = (0-0)\hat{i} - (0-0)\hat{j} + (\cos x \cos y - \cos x \cos y)\hat{k} = \vec{0}$$

$$\int \cos x \sin y\, dx = \sin x \sin y + f(y)$$

$$\int \sin x \cos y\, dy = \sin x \cos y + g(x)$$

$$\varphi = \sin x \sin y$$

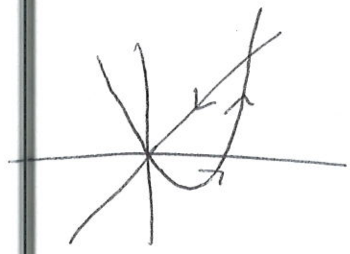
$$\int_C \cos x \sin y\, dx + \sin x \cos y\, dy = \sin\left(\tfrac{3\pi}{2}\right)\sin\left(\tfrac{\pi}{2}\right) - \sin(0)\sin(-\pi)$$

$$= (-1)(1) = \boxed{-1}$$

9. Use Green's Theorem to evaluate $\int_C \overset{M}{(y-x)}dx + \overset{N}{(2x-y)}dy$ on the path described by the boundary of the graphs $y = x, y = x^2 - 2x$ oriented counterclockwise. (12 points)

$$\frac{\partial N}{\partial x} = 2 \qquad \frac{\partial M}{\partial y} = 1 \qquad \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 2 - 1 = 1$$

$$\int_0^3 \int_{x^2-2x}^{x} 1 \, dy\, dx =$$



$$\int_0^3 x^2 - 3x \, dx = \frac{1}{3}x^3 - \frac{3}{2}x^2 \Big|_0^3 =$$

$$9 - \frac{27}{2} = \boxed{-\frac{9}{2}}$$

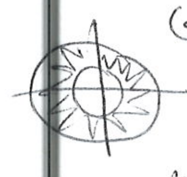
$x = x^2 - 2x$

$0 = x^2 - 3x$

$0 = x(x-3)$

$x = 0, x = 3$

10. Evaluate the surface integral $\int_S \int f(x,y,z)dS$ for $f(x,y,z) = \frac{xy}{z}$, $S: z = x^2 + y^2$, $4 \leq x^2 + y^2 \leq 16$. (12 points)



$G = x^2 + y^2 - z$

$\nabla G = \langle 2x, 2y, -1 \rangle$

$\|\nabla G\| = \sqrt{4x^2 + 4x^2 + 1}$

$= \sqrt{4r^2+1}$

$$\int_0^{2\pi} \int_2^4 \frac{\cancel{r^2} \sin\theta\cos\theta}{\cancel{r^2}} \sqrt{4r^2+1}\; r\, dr\, d\theta$$

$$\int_0^{2\pi} \int_2^4 \sin\theta\cos\theta \underline{r\sqrt{4r^2+1}\, dr}\, d\theta$$

$u = 4r^2 + 1$

$du = 8r\,dr$

$\frac{1}{8}du = r\,dr$

$r = 2 \to u = 17$

$r = 4 \to u = 65$

$\frac{1}{8}\int u^{1/2} du =$

$\frac{1}{48} \cdot \frac{2}{3} u^{3/2} \Big|_{17}^{65} = \frac{1}{12}(65^{3/2} - 17^{3/2})$

$$\int_0^{2\pi} \sin\theta\cos\theta \frac{1}{12}(65^{3/2} - 17^{3/2})\, d\theta$$

$$\frac{1}{2}\sin^2\theta \frac{1}{12}(65^{3/2} - 17^{3/2}) \Big|_0^{2\pi} = \boxed{0}$$

11. Evaluate the flux integral $\int_S \int \vec{F} \cdot \vec{N} dS$ where $\vec{F} = 3z\hat{\imath} - 4\hat{\jmath} + y\hat{k}$ for the surface $S: z = 1 - x - y$ in the first octant. (12 points)

$G = x + y + z - 1 = 0$ $\qquad\qquad$ $z = 0$

$\nabla G = \langle 1, 1, 1 \rangle$ $\qquad\qquad$ $y = 1 - x$

$$F \cdot \nabla G = \langle 3z, -4, y \rangle \cdot \langle 1, 1, 1 \rangle =$$

$$3z - 4 + y$$

$$3(1-x-y) - 4 + y$$

$$3 - 3x - 3y - 4 + y$$

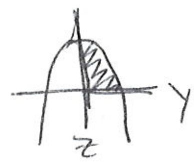
$$-1 - 3x - 2y$$

$$\int_0^1 \int_0^{1-x} -1 - 3x - 2y \, dy dx = \int_0^1 \cancel{-y} - 3xy - y^2 \Big|_0^{1-x} dx = \int_0^1 x - 1 - 3x(1-x) - \underbrace{(1-x)^2}_{1-2x+x^2} dx$$

$$= \int_0^1 \cancel{x} - 1 - \cancel{3x} + 3x^2 - 1 + \cancel{2x} - x^2 dx = \int_0^1 -2 + 2x^2 dx = -2x + \tfrac{2}{3}x^3 \Big|_0^1 = -2 + \tfrac{2}{3} = \boxed{-\tfrac{4}{3}}$$

12. Use the divergence theorem to evaluate $\int_S \int \vec{F} \cdot \vec{N} dS$ for $\vec{F}(x, y, z) = (4xy + z^2)\hat{\imath} + (2x^2 + 6yz)\hat{\jmath} + 2xz\hat{k}$ for the closed surface bounded by $x = 4, z = 9 - y^2$ and the coordinate planes. (14 points)

$$\vec{\nabla} \cdot \vec{F} = 4y + 6z + 2x$$



$$\int_0^4 \int_0^3 \int_0^{9-y^2} 4y + 6z + 2x \, dz dy dx$$

$$\int_0^4 \int_0^3 (9-y^2)(4y+2x) + 3(9-y^2)^2 \, dy dx =$$

$$36y + 18x - 4y^3 - 2xy^2 + 3(81 - 18y^2 + y^4)$$

$$\int_0^4 \int_0^3 36y + 18x - 4y^3 - 2xy^2 + 243 - 54y^2 + 3y^4 \, dy$$

$$\int_0^4 18y^2 + 18xy - y^4 - \tfrac{2}{3}xy^3 + 243y - 18y^3 + \tfrac{3}{5}y^5 \Big|_0^3$$

$$\int_0^4 54x + 18x + \frac{2349}{5} dx = \int_0^4 72x + \frac{2349}{5} dx =$$

$$36x^2 + \frac{2349}{5}x \Big|_0^4 = \frac{12{,}276}{5} = 2455.2$$

13. Evaluate $\int_C \vec{F} \cdot d\vec{r}$ using Stokes' Theorem for $\vec{F}(x, y, z) = xyz\hat{\imath} + y\hat{\jmath} + z\hat{k}$ for $S: z = x^2$, $0 \le x \le 2, 0 \le y \le 2$. (12 points)

$$\nabla \times F = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xyz & y & z \end{vmatrix} =$$

$$(0-0)\hat{\imath} - (0-xy)\hat{\jmath} + (0-xz)\hat{k}$$

$$G = z - x^2$$

$$\nabla G = \langle -2x, 0, 1 \rangle$$

$$(\vec{\nabla} \times \vec{F}) \cdot \nabla G = \langle 0, xy, -xz \rangle \cdot \langle -2x, 0, 1 \rangle =$$

$$0 + 0 - xz = x(x^2) = x^3$$

$$z = x^2$$

$$\int_0^2 \int_0^2 x^3 \, dy \, dx = \int_0^2 2x^3 \, dx = \frac{1}{2} x^4 \Big|_0^2 = \boxed{8}$$