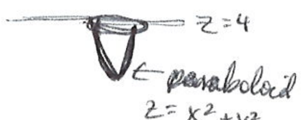
MTH 277, Exam #2, Spring 2018 Name KEY

Instructions: Show all work. Give exact answers unless specifically asked to round. All complex numbers should be stated in standard form, and all complex fractions should be simplified. If you do not show work, problems will be graded as "all or nothing" for the answer only; partial credit will not be possible and any credit awarded for the work will not be available.

1. Convert the triple integrals to the given coordinate systems and then complete the integration. Describe the region being integrated (over). (10 points each)

   a. $\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{x^2+y^2}^{4} x\,dz\,dy\,dx$ in cylindrical.

$$\int_0^{2\pi}\int_0^2\int_{r^2}^4 r\cos\theta\cdot r\,dz\,dr\,d\theta$$



$$\int_0^{2\pi}\int_0^2\int_{r^2}^4 r^2\cos\theta\,dz\,dr\,d\theta =$$

$$\int_0^{2\pi}\int_0^2 r^2\cos\theta(4-r^2)\,dr\,d\theta = \int_0^{2\pi}\int_0^2 4r^2\cos\theta - r^4\cos\theta\,dr\,d\theta =$$

$$\int_0^{2\pi} \frac{4}{3}r^3\cos\theta - \frac{r^5}{5}\cos\theta\Big|_0^2 d\theta = \int_0^{2\pi}\left(\frac{32}{3}-\frac{32}{5}\right)\cos\theta\,d\theta = \left(\frac{32}{3}-\frac{32}{5}\right)\sin\theta\Big|_0^{2\pi}$$

$$=0$$

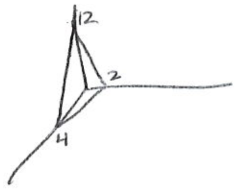
   b. $\int_0^3\int_0^{\sqrt{9-x^2}}\int_0^{\sqrt{9-x^2-y^2}}\sqrt{x^2+y^2+z^2}\,dz\,dy\,dx$ in spherical.

$$\int_0^{\pi/2}\int_0^{\pi/2}\int_0^3 \rho^3\sin\varphi\,d\rho\,d\theta\,d\varphi = \int_0^{\pi/2}\int_0^{\pi/2}\frac{1}{4}\rho^4\Big|_0^3 d\theta\,\sin\varphi\,d\varphi =$$

$$\int_0^{\pi/2}\int_0^{\pi/2}\frac{81}{4}\sin\varphi\,d\theta\,d\varphi = \int_0^{\pi/2}\frac{81\pi}{8}\sin\varphi\,d\varphi = -\frac{81\pi}{8}\cos\varphi\Big|_0^{\pi/2} =$$

$$-0+\frac{81\pi}{8} = \boxed{\frac{81\pi}{8}}$$

2. Find the volume bounded by the coordinate axes and the plane $3x + 6y + z = 12$. (10 points)



$$\int_0^4 \int_0^{2-\frac{1}{2}x} \int_0^{-3x-6y+12} dz\,dy\,dx$$

$\frac{12-3x}{6} = y$

$2-\frac{1}{2}x$

$$\int_0^4 \int_0^{2-\frac{1}{2}x} -3x-6y+12\,dy\,dx =$$

$$\int_0^4 -3x\left(2-\tfrac{1}{2}x\right) - 3\left(2-\tfrac{1}{2}x\right)^2 + 12\left(2-\tfrac{1}{2}x\right)dx =$$

$$-3\left(4-2x+\tfrac{1}{4}x^2\right)$$

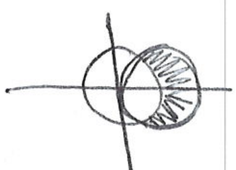
$$\int_0^4 \frac{3x^2}{4} - 6x + 12\,dx =$$

$$\frac{x^3}{4} - 3x^2 + 12x \Big|_0^4 = 16$$

3. Set up a double integral to find the area inside the circle $r = 2\cos\theta$ and outside $r = 1$. Do not integrate. (8 points)

$1 = 2\cos\theta$

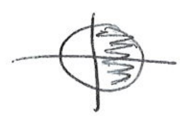
$\frac{1}{2} = \cos\theta$



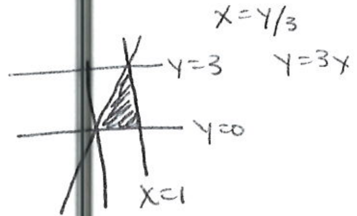
$$\int_{-\pi/3}^{\pi/3} \int_1^{2\cos\theta} r\,dr\,d\theta$$

4. Set up a double integral to find the volume under $f(x,y) = e^{-(x^2+y^2)/2}$ on the region $x^2 + y^2 \le 25, x \ge 0$. Do not integrate. (8 points)

$$\int_{-\pi/2}^{\pi/2} \int_0^5 e^{-r^2/2}\,r\,dr\,d\theta$$



5. Change the order of integration in $\int_0^3 \int_{y/3}^1 \frac{1}{1+x^4}\,dx\,dy$ so that it can be integrated. Then complete the integration. (10 points)



$$\int_0^1\int_0^{3x} \frac{1}{1+x^4}\,dy\,dx =$$

$$\int_0^1 \frac{3x}{1+x^4}\,dx$$

$u = x^2$
$du = 2x\,dx$
$\frac{1}{2}du = x\,dx$

$$\frac{3}{2}\int_0^1 \frac{1}{1+u^2}\,du = \frac{3}{2}\arctan x^2 \Big|_0^1 = \frac{3}{2}\left(\frac{\pi}{4}\right) = \boxed{\frac{3\pi}{8}}$$

6. Determine if the vector field $\vec{F}(x,y,z) = y^2z\hat{\imath} + 2xyz\hat{\jmath} + xy^2\hat{k}$ is conservative. Find the potential function. (10 points)

$$\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y^2z & 2xyz & xy^2 \end{vmatrix} = (2xy - 2xy)\hat{\imath} - (y^2 - y^2)\hat{\jmath} + (2yz - 2yz)\hat{k}$$

$$= \vec{0}$$

Conservative

$$\int y^2 z\,dx = xy^2z + f(y,z)$$

$$\int 2xyz\,dy = xy^2z + g(x,z)$$

$$\int xy^2\,dz = xy^2z + h(x,y)$$

$$\varphi = xy^2z + K$$

7. Evaluate the line integral $\int_C \vec{F} \cdot d\vec{r}$ for $\vec{F}(x,y,z) = x^2\hat{\imath} + y^2\hat{\jmath} + z^2\hat{k}$ on the path $C: \vec{r}(t) = 2\sin t\,\hat{\imath} + 2\cos t\,\hat{\jmath} + \frac{1}{2}t^2\hat{k}$ on $[0,\pi]$. (10 points)

$F(t) = 4\sin^2 t\,\hat{\imath} + 4\cos^2 t\,\hat{\jmath} + \frac{1}{4}t^4\hat{k}$

$r'(t) = 2\cos t\,\hat{\imath} - 2\sin t\,\hat{\jmath} + 2t\,\hat{k}$

$F \cdot dr = 8\sin^2 t\cos t - 8\cos^2 t\sin t + \frac{1}{2}t^5$

$\int_0^\pi 8\sin^2 t\cos t - 8\cos^2 t\sin t + \frac{1}{2}t^5\,dt$

$\cancel{\frac{8}{3}\sin^3 t}^{\,0} + \frac{8}{3}\cos^3 t + \frac{1}{12}t^6\Big|_0^\pi = -\frac{16}{3} + \frac{\pi^6}{12}$

8. Evaluate the line integral $\int_C 2xyz\,ds$ on the path $\vec{r}(t) = t\hat{\imath} + 5t\hat{\jmath} + 3t\hat{k}$ on $[0,1]$. (8 points)

$r'(t) = \langle 1, 5, 3\rangle$

$\|r'(t)\| = \sqrt{1+25+9} = \sqrt{35}$

$\int_0^1 2(t)(5t)(3t)\sqrt{35}\,dt =$

$30\sqrt{35}\int_0^1 t^3\,dt = \frac{30\sqrt{35}}{4}t^4\Big|_0^1 = \boxed{\frac{30\sqrt{35}}{4}}$

9. Match the vector field equations to the appropriate graph. (3 points each)

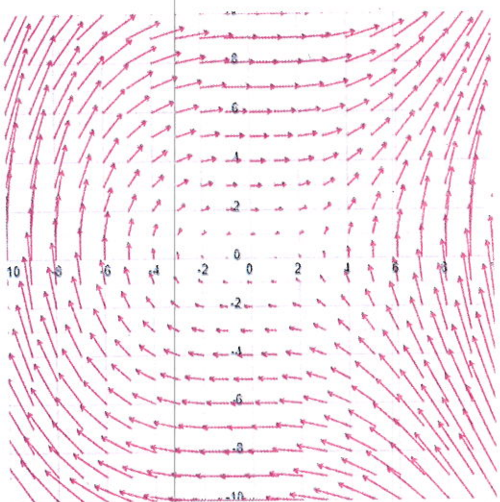
a. $\vec{F}(x,y) = \sin x\,\hat{\imath} + \cos y\,\hat{\jmath}$ iii

b. $\vec{F}(x,y) = 3x\hat{\imath} - 2y\hat{\jmath}$ ii

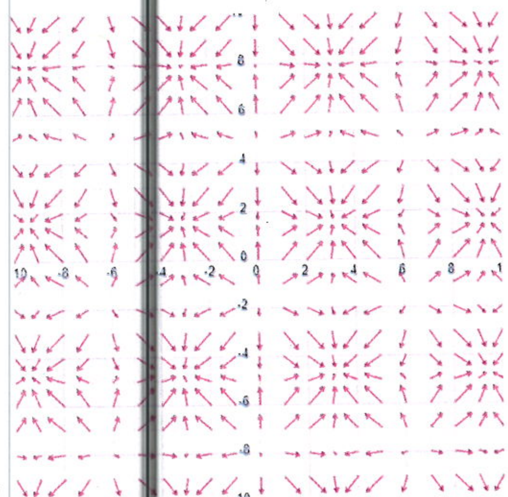
c. $\vec{F}(x,y) = \ln(x^2+y^2)\,\hat{\imath} + xy\hat{\jmath}$ iv

d. $\vec{F}(x,y) = y\hat{\imath} + \frac{x^2}{2}\hat{\jmath}$ i

d



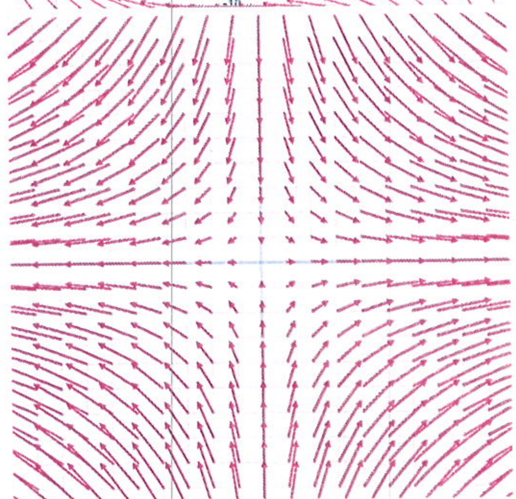
i.



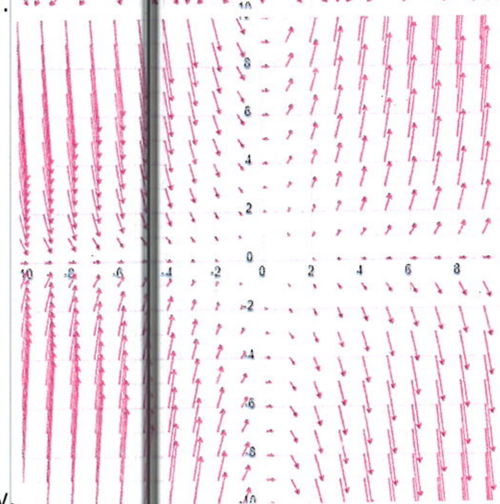
iii.

a

b



ii.



iv.

c

10. Find ∇f and $\nabla^2 f$ for $f(x,y,z) = 2z^2 y \sin x$. (8 points)

$\nabla f = \langle 2z^2 y\cos x,\ 2z^2 \sin x,\ 4yz \sin x \rangle$

$\nabla^2 f =$

$-2z^2 y \sin x + 0 + 4y \sin x$

11. Find $\nabla \cdot \vec{F}$ and $\nabla \times \vec{F}$ for $\vec{F}(x, y, z) = xz\hat{\imath} + yz\hat{\jmath} + xyz\hat{k}$. (10 points)

$$\vec{\nabla} \cdot \vec{F} = z + z + xy = 2z + xy$$

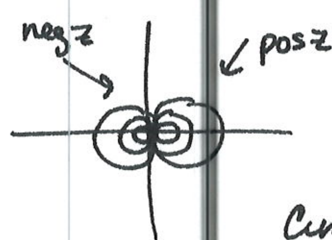
$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xz & yz & xyz \end{vmatrix} = (xz - y)\hat{\imath} - (yz - x)\hat{\jmath} + (0 - 0)\hat{k}$$

12. Sketch the level curves for $f(x, y) = \frac{x}{x^2+y^2}$ for values of z in $[-2,2]$. (8 points)

$$x^2 + y^2 = \frac{x}{z}$$

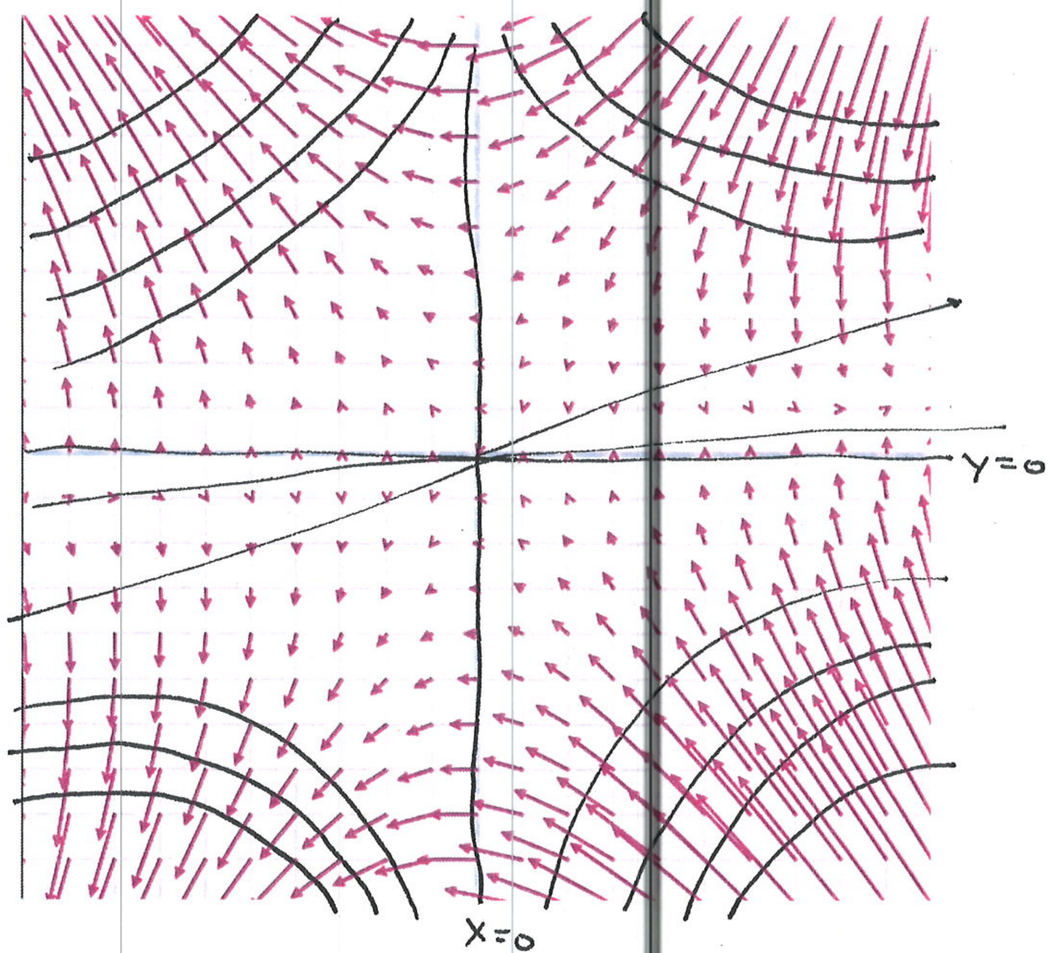
$$r^{\cancel{2}} = \frac{\cancel{r}\cos\theta}{z}$$

$$r = \frac{\cos\theta}{z}$$



Circle grows as $z \to 0$

13. Drawn below is the gradient field for $z = 2x^2y - 8xy^2$ using nullclines. Use that information to sketch several level curves of the function. (10 points)



$\nabla f = \langle 4xy - 8y^2, 2x^2 - 16xy \rangle$

$4xy - 8y^2 = 0$

$4y(x - 2y) = 0$

$y = 0, \ y = \frac{1}{2}x$

$2x^2 - 16xy = 0$

$2x(x - 8y) = 0$

$x = 0, \ y = \frac{1}{8}x$