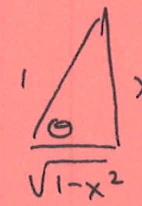
MTH 174 Homework #3 Key

1a. $\int x^3\sqrt{1-x^2}\,dx$ $\qquad x = \sin\theta$

$dx = \cos\theta\, d\theta$

$\int \sin^3\theta \cos\theta \cdot \cos\theta\, d\theta \qquad \sqrt{1-x^2} = \cos\theta$



$\int \sin\theta \sin^2\theta \cos^2\theta\, d\theta$

$\int \sin\theta(1-\cos^2\theta)\cos^2\theta\, d\theta \qquad u = \cos\theta$

$du = -\sin\theta\, d\theta$

$-\int (1-u^2)u^2\, du$

$+\int u^4 - u^2\, du = \frac{1}{5}[u^5] - \frac{1}{3}u^3 + C$

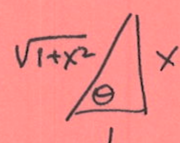
$\frac{1}{5}\cos^5\theta - \frac{1}{3}\cos^3\theta + C$

$$\boxed{\frac{1}{5}(1-x^2)^{5/2} - \frac{1}{3}(1-x^2)^{3/2} + C}$$

b. $\int \frac{\sqrt{1+x^2}}{x}\,dx$ $\qquad x = \tan\theta$

$dx = \sec^2\theta\, d\theta$

$\sqrt{1+x^2} = \sec\theta$



$\int \frac{\sec\theta \sec^2\theta\, d\theta}{\tan\theta} = \int \frac{\sec\theta\, \cancel{\sec^2}\theta\, \tan\theta}{\tan\theta \cdot \tan\theta}\, d\theta = \int \frac{\sec^2\theta \quad \sec\theta\tan\theta\, d\theta}{1+\sec^2\theta}$

$u = \sec\theta$

$du = \sec\theta\, d\theta \tan\theta$

$\int \frac{u^2\, du}{1+u^2} = \int 1 - \frac{1}{u^2+1}\, du$

$$\begin{array}{r} 1 \\ u^2+1\,\overline{)\,u^2} \\ -u^2 \mp 1 \\ \hline -1 \end{array}$$

$u - \arctan u + C$

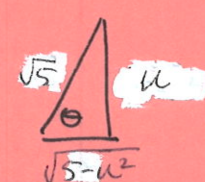
$\sec\theta - \arctan(\sec\theta) + C$

$$\boxed{\sqrt{1+x^2} - \arctan(\sqrt{1+x^2}) + C}$$

c. $\int \frac{du}{u\sqrt{5-u^2}}$ $\qquad u = \sqrt{5}\sin\theta$

$du = \sqrt{5}\cos\theta\, d\theta$

$\sqrt{5-u^2} = \sqrt{5}\cos\theta$



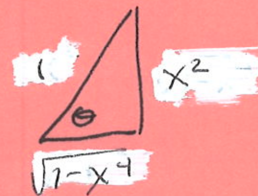
$\int \frac{\sqrt{5}\cos\theta\, d\theta}{\sqrt{5}\sin\theta \cdot \sqrt{5}\cos\theta} = \frac{1}{\sqrt{5}}\int \csc\theta\, d\theta = -\frac{1}{\sqrt{5}}\ln|\csc\theta + \cot\theta| + C$

$$\boxed{-\frac{1}{\sqrt{5}}\ln\left|\frac{\sqrt{5}}{u} + \frac{\sqrt{5-u^2}}{u}\right| + C}$$

1d. $\int x\sqrt{1-x^4}\,dx$ $\quad x^2 = \sin\theta$

$2x\,dx = \cos\theta\,d\theta$

$\frac{1}{2}\int \cos\theta\cos\theta\,d\theta \quad \sqrt{1-x^4} = \cos\theta$



$\frac{1}{2}\int \frac{1}{2}(1+\cos 2\theta)\,d\theta = \frac{1}{4}\left[\theta + \frac{1}{2}\sin 2\theta\right] + C$

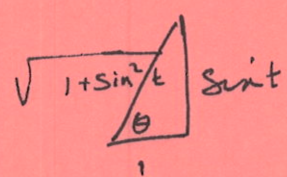
$\frac{1}{4}\theta + \frac{1}{8}\cdot 2\sin\theta\cos\theta + C = \frac{1}{4}\theta + \frac{1}{4}\sin\theta\cos\theta + C$

$= \boxed{\frac{1}{4}\arcsin x^2 + \frac{1}{4}\cdot x^2\sqrt{1-x^4} + C}$

e. $\int_0^{\pi/2} \frac{\cos t}{\sqrt{1+\sin^2 t}}$ $\quad \sin t = \tan\theta \rightarrow {}^{1}_{0}$

$\cos t\,dt = \sec^2\theta\,d\theta$

$\sqrt{1+\cancel{\tan}^2 t} = \sec\theta$



$\int_0^1 \frac{\sec^{\cancel{2}}\theta\,d\theta}{\cancel{\sec\theta}}$

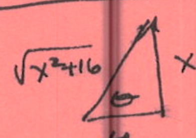
$\int_0^1 \sec\theta\,d\theta = \ln|\sec\theta + \tan\theta|\Big|_0^1 = \ln|\sec 1 + \tan 1| - \ln|\cancel{1} + 0|$

$= \boxed{\ln|\sec 1 + \tan 1|}$

f. $\int \frac{dx}{\sqrt{x^2+16}}$ $\quad x = 4\tan\theta$

$dx = 4\sec^2\theta\,d\theta$

$\sqrt{x^2+16} = 2\sec\theta$



$\int \frac{\overset{2}{\cancel{4}}\sec^{\cancel{2}}\theta\,d\theta}{\cancel{2}\cancel{\sec\theta}} = \int 2\sec\theta\,d\theta = 2\ln|\sec\theta + \tan\theta| + C$

$\boxed{2\ln\left|\frac{\sqrt{x^2+16}}{4} + \frac{x}{4}\right| + C}$

g. $\int \frac{x}{\sqrt{x^2-7}}\,dx$ $\quad x = \sqrt{7}\sec\theta$

$dx = \sqrt{7}\sec\theta\tan\theta\,d\theta$

$\sqrt{x^2-7} = \sqrt{7}\tan\theta$

$\int \frac{\sqrt{7}\sec\theta\cdot\cancel{\sqrt{7}}\sec\theta\,\cancel{\tan\theta}\,d\theta}{\cancel{\sqrt{7}}\cancel{\tan\theta}} = \int \sqrt{7}\sec^2\theta\,d\theta = \sqrt{7}\tan\theta + C$

$= \boxed{\sqrt{x^2-7}\cdot\sqrt{7} + C}$

1h. $\int \frac{x}{\sqrt{x^2+x+1}}\,dx = \int \frac{x}{\sqrt{x^2+x+\frac14+\frac34}}\,dx = \int \frac{x}{\sqrt{(x+1/2)^2+\frac34}}\,dx$

$u = x + 1/2$

$u - 1/2 = x$

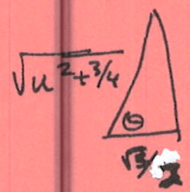
$du = dx$

$\int \frac{u-1/2}{\sqrt{u^2+3/4}}\,du$

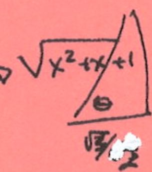
$u = \frac{\sqrt3}{2}\tan\theta$

$du = \frac{\sqrt3}{2}\sec^2\theta\, d\theta$

$\sqrt{u^2+3/4} = \frac{\sqrt3}{2}\sec\theta$



$\Longrightarrow$



$\int \frac{\frac{\sqrt3}{2}\tan\theta - 1/2}{\frac{\sqrt3}{2}\cancel{\sec\theta}}\,\frac{\sqrt3}{2}\sec^{\cancel{2}}\theta\, d\theta$

$\int \frac{\sqrt3}{2}\sec\theta\tan\theta - 1/2\sec\theta\, d\theta = \frac{\sqrt3}{2}\sec\theta - \frac12\ln|\sec\theta+\tan\theta| + C$

$\frac{\sqrt3}{2}\cdot\frac{\sqrt{x^2+x+1}}{\sqrt3/2} - \frac12\ln\left|\frac{\sqrt{x^2+x+1}}{\sqrt3/2} + \frac{x+1/2}{\sqrt3/2}\right| + C$

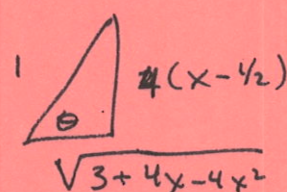
$= \boxed{\sqrt{x^2+x+1} - \frac12\ln\left|\frac{2}{\sqrt3}\left(\sqrt{x^2+x+1} + x + 1/2\right)\right| + C}$

i. $\int \frac{x^2}{(3+4x-4x^2)^{3/2}}\,dx$

$3 + (\ 4x - 4x^2) = 3 - (\frac14 - x + x^2)4 + 1$

$\int \frac{x^2}{\left(4-[2(x-1/2)]^2\right)^{3/2}}\,dx$

$\sin\theta = 2(x-1/2)\cdot 2$



$\frac14\sin\theta = x - 1/2$

$\frac12(\sin\theta+1) = \frac12\sin\theta + \frac12 = x$

$\frac14\cos\theta\, d\theta = dx$

$x^2 = \frac14(\sin^2\theta + 2\sin\theta + 1)$

$\sqrt{4-[2(x-1/2)]^2} = 2\cos\theta$

$\int \frac{\frac14(\sin^2\theta + 2\sin\theta + 1)}{(2\cos\theta)^3}\cdot\frac14\cos\theta\, d\theta$

$\int \frac{1}{128}\cdot\frac{\sin^2\theta + 2\sin\theta + 1}{\cos^2\theta}\, d\theta = \frac{1}{128}\int \tan^2\theta + 2\sec\theta\tan\theta + \sec^2\theta\, d\theta =$

$\frac{1}{128}\int (\sec^2\theta - 1) + 2\sec\theta\tan\theta + \sec^2\theta\, d\theta =$

$\frac{1}{128}\int 2\sec^2\theta + 2\sec\theta\tan\theta - 1\, d\theta = \frac{1}{128}\left[2\tan\theta + 2\sec\theta - \theta\right] + C$

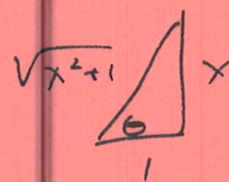
$\boxed{\frac{1}{64}\cdot\frac{4(x-1/2)}{\sqrt{3+4x-4x^2}} + \frac{1}{64}\frac{1}{\sqrt{3+4x-4x^2}} + \arcsin(4(x-1/2)) + C}$

1j. $\int x^2(x^2+1)^{3/2}\,dx$

$x = \tan\theta$

$dx = \sec^2\theta\,d\theta$

$\sqrt{x^2+1} = \sec\theta$



$\int \tan^2\theta \sec^2\theta\,d\theta \sec^3\theta$

$\int \tan^2\theta \sec^5\theta\,d\theta$ → integrate w/ wolfram alpha if you like

$\frac{19}{128}\sin\theta\sec^6\theta - \frac{17}{768}\sin 3\theta\sec^6\theta - \frac{1}{256}\sin 5\theta\sec^6\theta + \frac{1}{16}\ln|\sec\theta+\tan\theta| + C =$

$\frac{19}{128}\sin\theta\sec^6\theta - \frac{17}{768}\left[3\cos^2\theta\sin\theta - \sin^3\theta\right]\sec^6\theta - \frac{1}{256}\left[5\cos^4\theta\sin\theta -\right.$

$\left.10\cos^2\theta\sin^3\theta + \sin^5\theta\right] + \frac{1}{16}\ln|\sec\theta+\tan\theta| + C$

$= \frac{19}{128}\sin\theta\sec^6\theta - \frac{17}{256}\sin\theta\sec^4\theta - \frac{17}{768}\sin^3\theta\sec^6\theta - \frac{5}{256}\sec^2\theta\sin\theta + \frac{5}{128}\sec^4\theta\sin^3\theta$

$- \frac{1}{256}\sin^5\theta\sec^6\theta + \frac{1}{16}\ln|\sec\theta+\tan\theta| + C$

$= \frac{19}{128}\,\frac{x}{\cancel{\sqrt{x^2+1}}}\cdot\frac{(\sqrt{x^2+1})^{\cancel{6}\,5}}{1} - \frac{17}{256}\cdot\frac{x}{\cancel{\sqrt{x^2+1}}}\cdot\frac{(\sqrt{x^2+1})^{\cancel{4}\,3}}{1} - \frac{17}{768}\,\frac{x^3}{\cancel{(\sqrt{x^2+1})^3}}\cdot\frac{(\sqrt{x^2+1})^{\cancel{6}\,3}}{1} -$

$\frac{5}{256}\cdot\frac{(\sqrt{x^2+1})^{\cancel{2}}}{1}\cdot\frac{x}{\cancel{\sqrt{x^2+1}}} + \frac{5}{128}\,\frac{(\sqrt{x^2+1})^{\cancel{4}}}{1}\cdot\frac{x^3}{\cancel{(\sqrt{x^2+1})^3}} - \frac{1}{256}\,\frac{x^5}{\cancel{(\sqrt{x^2+1})^5}}\cdot\frac{(\sqrt{x^2+1})^{\cancel{6}}}{1}$

$+ \frac{1}{16}\ln\left|\sqrt{x^2+1}+x\right| + C$

$$= \boxed{\frac{19}{128}x(x^2+1)^{5/2} - \frac{17}{256}x(x^2+1)^{3/2} - \frac{17}{768}x^3(x^2+1)^{3/2} - \frac{5}{256}x\sqrt{x^2+1} + \frac{5}{128}x^3\sqrt{x^2+1} - \frac{1}{256}x^5\sqrt{x^2+1} + \frac{1}{16}\ln\left|\sqrt{x^2+1}+x\right| + C}$$

2a. $\int \frac{1+6x}{(4x-3)(2x+5)}\,dx = \int \frac{A}{4x-3} + \frac{B}{2x+5}\,dx = \frac{11}{13}\int\frac{1}{4x-3}\,dx + \frac{14}{13}\int\frac{1}{2x+5}\,dx$

$2Ax + 5A + 4Bx - 3B = 1+6x$

$2A + 4B = 6$

$5A - 3B = 1$

$A = 11/13 \quad B = 14/13$

$= \frac{11}{13}\cdot\frac{1}{4}\ln|4x-3| + \frac{14}{13}\cdot\frac{1}{2}\ln|2x+5| + C$

$$= \boxed{\frac{11}{52}\ln|4x-3| + \frac{7}{13}\ln|2x+5| + C}$$

2b. $\int \frac{1}{(x^2-9)^2}\,dx = \int \frac{1}{(x-3)^2(x+3)^2}\,dx$

$\int \frac{A}{x-3} + \frac{B}{(x-3)^2} + \frac{C}{x+3} + \frac{D}{(x+3)^2}\,dx$

$A(x-3)(x+3)^2 + B(x+3)^2 + C(x+3)(x-3)^2 + D(x-3)^2 = 1$

$Ax^3 + A3x^2 - A9x - 27A + Bx^2 + 6Bx + 9B + Cx^3 - 3Cx^2 - 9Cx + 27C$
$+ Dx^2 - 6Dx + 9D = 1$

$A + C = 0$
$3A + B - 3C + D = 0$
$-9A + 6B - 9C - 6D = 0$
$-27A + 9B + 27C + 9D = 1$

$A = -1/108$
$B = 1/36$
$C = 1/108$
$D = 1/36$

$-\frac{1}{108}\int \frac{1}{x-3}\,dx + \frac{1}{36}\int \frac{1}{(x-3)^2}\,dx + \frac{1}{108}\int \frac{1}{x+3}\,dx + \frac{1}{36}\int \frac{1}{(x+3)^2}\,dx$

$\boxed{-\frac{1}{108}\ln|x-3| - \frac{1}{36}\cdot\frac{1}{(x-3)} + \frac{1}{108}\ln|x+3| - \frac{1}{36}\cdot\frac{1}{(x+3)} + C}$

2c. $\int \frac{t^6+1}{t^6+t^3}\,dt$

$$t^6+t^3 \overline{\big)\, t^6 \quad +1} \quad 1$$
$-t^3 \mp t^3$
$-t^3+1$

$\int 1 + \frac{1-t^3}{t^6+t^3}\,dt$

$(t^6+t^3) = t^3(t^3+1)$

$t^3(t+1)(t^2-t+1)$

$\frac{A}{t} + \frac{B}{t^2} + \frac{C}{t^3} + \frac{D}{t+1} + \frac{Et+F}{t^2-t+1}$

$At^2(t^3+1) + Bt(t^3+1) + C(t^3+1) + Dt^3(t^2-t+1) + (Et+F)(t+1)t^3 = 1-t^3$

$At^5 + At^2 + Bt^4 + Bt + Ct^3 + C + Dt^5 - Dt^4 + Dt^3 + Et^5 + Et^4 + Ft^4 + Ft^3 = 1-t^3$

t^5 $A + D + E = 0$ $\quad D + E = 0$
t^4 $B - D + E = 0$ $\quad \cancel{B} - D + E = 0$
t^3 $C + D + F = -1$ $\quad 1 + D + F = -1$
t^2 $A = 0$
1 $C = 1$
t $B = 0$

$D + E = 0 \quad E = 0$
$-D + E = 0 \quad D = 0$
$D + F = -2 \quad F = -2$

$(Et+F)(t+1) =$
$Et^2 + Et + Ft + F$

2c cont'd

$C=1,\ F=-2$

$$\int 1+\frac{1}{t^3}-\frac{2}{t^2-t+1}\,dt=\int 1+t^{-3}-\frac{2}{t^2-t+\frac{1}{4}+\frac{3}{4}}\,dt$$

$$(t-1/2)^2+\tfrac{3}{4}$$

$$=t-\frac{1}{t^2}-2\cdot\frac{2}{\sqrt{3}}\arctan\left(\frac{t-1/2}{\frac{\sqrt{3}}{2}}\right)+C$$

$$\boxed{t-\frac{1}{t^2}-\frac{4}{\sqrt{3}}\arctan\left(\frac{2t-1}{\sqrt{3}}\right)+C}$$

2d. $\displaystyle\int \frac{x^5+x-1}{x^3+1}\,dx$

$$\begin{array}{r}x^2\\ x^3+1\,\overline{)\,x^5+0x^4+0x^3+0x^2+x-1}\\ -\ x^5\qquad\qquad\mp x^2\\ \hline -x^2+x-1\end{array}$$

$x^3+1=(x+1)(x^2-x+1)$

$$\int x^2-\frac{x^2-x+1}{x^3+1}\,dx=\int x^2-\frac{1}{x+1}\,dx$$

$$\boxed{\frac{1}{3}x^3-\ln|x+1|+C}$$

2e. $\displaystyle\int\frac{e^{2x}}{e^{2x}+3e^x+2}\,dx$ $\quad u=e^x$, $du=e^x$ $\quad\displaystyle\int\frac{u}{u^2+3u+2}\,du=\int\frac{u\,du}{(u+2)(u+1)}$

$$\int\frac{A}{u+2}+\frac{B}{u+1}\,du$$

$Au+A+Bu+2B=u$

$A+B=1$ $\quad A=2$

$A+2B=0$ $\quad B=-1$

$$\int\frac{2}{u+2}-\frac{1}{u+1}\,du$$

$$=2\ln|u+2|-\ln|u+1|+C=\boxed{2\ln|e^x+2|-\ln|e^x+1|+C}$$

f. $\displaystyle\int\frac{x^4+1}{x^5+4x^3}\,dx=\int\frac{x^4+1}{x^3(x^2+4)}\,dx$ $\quad\displaystyle\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x^3}+\frac{Dx+E}{x^2+4}$

$Ax^4+A4x^2+Bx^3+B4x+Cx^2+C4+Dx^4+Ex^3=x^4+1$

$A+D=1$ $\quad C=1/4$ $\quad 4A+1/4=0$

$B+E=0$ $\quad B=0$ $\quad 4A=-1/4$

$4A+C=0$ $\quad E=0$ $\quad A=-1/16$

$4B=0$

$4C=1$ $\quad D=1+1/16=17/16$

$$\int\frac{-\frac{1}{16}}{x}\,dx+\int\frac{1}{4}x^{-3}\,dx+\int\frac{17/16\,x}{x^2+4}\,dx=\boxed{-\frac{1}{16}\ln|x|-\frac{1}{8x^2}+\frac{17}{32}\ln|x^2+4|+C}$$

2g. $\displaystyle\int \frac{x^4-2x^3+x^2+2x-1}{x^2-2x+1}\,dx$

$$\begin{array}{r|l} & x^2 \\ x^2-2x+1 & x^4-2x^3+x^2+2x-1 \\ & -\ x^4 \pm 2x^3 \mp x^2 \\ \hline & 2x-1 \end{array}$$

$\displaystyle\int x^2 + \frac{2x-1}{x^2-2x+1}\,dx = \int x^2 + \frac{2x-1}{(x+1)^2}\,dx$

$\displaystyle\frac{A}{x+1} + \frac{B}{(x+1)^2} \Rightarrow$ $Ax+A+B = 2x-1$

$A = 2$

$A+B=-1 \rightarrow 2+B=-1 \rightarrow B=-3$

$\displaystyle\int x^2 + \frac{2}{x+1} - \frac{3}{(x+1)^2}\,dx = \boxed{\frac{1}{3}x^3 + 2\ln|x+1| + \frac{3}{x+1} + C}$

h. $\displaystyle\int \frac{4x}{x^3+x^2+x+1}\,dx = \int \frac{4x}{x^2(x+1)+1(x+1)}\,dx = \int \frac{4x}{(x^2+1)(x+1)}\,dx$

$\displaystyle\frac{Ax+B}{x^2+1} + \frac{C}{x+1} \Rightarrow$ $(Ax+B)(x+1) + Cx^2+C = 4x$

$Ax^2+Ax+Bx+B+Cx^2+C=4x$

$Ax^2+Cx^2=0 \rightarrow A+C=0$

$A+B=4$

$B+C=0$ $\qquad A=2,\ B=2,\ C=-2$

$\displaystyle\int \frac{2x}{x^2+1} + \frac{2}{x^2+1} - \frac{2}{x+1}\,dx = \boxed{\ln|x^2+1| + 2\arctan(x) - 2\ln|x+1| + C}$

i. $\displaystyle\int \frac{x^3+2x}{x^4+4x^2+3}\,dx = \int \frac{x^3+2x}{(x^2+3)(x^2+1)}\,dx$ $\qquad \displaystyle\frac{Ax+B}{x^2+3} + \frac{Cx+D}{x^2+1} \Rightarrow$

$Ax^3+Bx^2+Ax+B+Cx^3+Dx^2+3Cx+3D = x^3+2x$

$A+C=1 \Rightarrow A=1/2 \quad B=0$

$B+D=0 \quad C=1/2 \quad D=0$

$A+3C=2$

$B+3D=0$

$$\int \frac{1/2}{x^2+3} + \frac{1/2}{x^2+1}\,dx = \frac{1}{2}\left(\frac{1}{\sqrt{3}}\arctan\left(\frac{x}{\sqrt{3}}\right)\right) + \frac{1}{2}\arctan x + C$$

$$\boxed{\frac{1}{2\sqrt{3}}\arctan\left(\frac{x}{\sqrt{3}}\right) + \frac{1}{2}\arctan x + C}$$

3a. $\int \frac{A}{x+3} + \frac{B}{(x+3)^2} + \frac{C}{x-2} + \frac{Dx+E}{x^2+4} + \frac{F}{x} + \frac{G}{x^2} + \frac{H}{x^3}\,dx$

b. $\int \frac{A}{x-2} + \frac{B}{x+2} + \frac{Cx+D}{x^2+7} + \frac{Ex+F}{(x^2+7)^2} + \frac{G}{x-1} + \frac{H}{(x-1)^2} + \frac{I}{(x-1)^3} + \frac{J}{x+1}\,dx$

c. $\int \frac{Ax+B}{x^2+2x+2} + \frac{C}{x+1} + \frac{D}{x+3} + \frac{E}{(x+1)^2} + \frac{F}{(x+3)^2} + \frac{G}{(x-1)} + \frac{H}{x+2} +$

$\frac{I}{(x+2)^2} + \frac{J}{(x+2)^3} + \frac{K}{(x+2)^4} + \frac{L}{(x+2)^5}\,dx$

4a. $\int \frac{\sqrt{x+1}}{x}\,dx \qquad u=\sqrt{x+1} \Rightarrow u^2 = x+1 \qquad \rightarrow \int \frac{u\cdot 2u\,du}{u^2-1}$

$2u\,du = dx \qquad x = u^2-1$

b. $\int \frac{x^3}{\sqrt[3]{x^2+1}}\,dx \qquad u = \sqrt[3]{x^2+1} \qquad \int \frac{(u^3-1)\left(\frac{3}{2}u^2\,du\right)}{u}$

$u^3 = x^2+1 \rightarrow u^3-1 = x^2$

$\int \frac{x^2\cdot x\,dx}{\sqrt[3]{x^2+1}} \qquad \frac{3u^2\,du}{2} = \cancel{2}x\,dx$

c. $\int \frac{1}{\sqrt{x}+\sqrt[3]{x}}\,dx \qquad u = x^{1/6} \qquad \int \frac{6u^5\,du}{u^3+u^2}$

$u^2 = x^{1/3}$

$u^3 = x^{1/2}$

$u^6 = x$

$6u^5\,du = dx$

4d. $\int \frac{dx}{1+e^x} dx \quad \frac{e^{-y}}{e^{-x}} \rightarrow \int \frac{e^{-x}}{e^{-x}+1} dx \quad u = e^{-x} \quad du = -e^{-x} \rightarrow \int \frac{-1}{u+1} du$ ⑨

e. $\int \frac{1}{1+\sqrt[3]{x}} dx$

$u = \sqrt[3]{x}$

$u^3 = x$

$3u^2 du = dx$

$\rightarrow \int \frac{3u^2\, du}{1+u}$

f. $\int \frac{\sqrt{x}}{x^2+x} dx$

$u = \sqrt{x} \quad 2u\, du = dx$

$u^2 = x$

$u^4 = x^2$

$\longrightarrow \int \frac{u \cdot 2u\, du}{u^4+u^2}$

g. $\int \frac{\sqrt{1+\sqrt{x}}}{x} dx$

$u = \sqrt{1+\sqrt{x}}$

$u^2 = 1+\sqrt{x}$

$u^2-1 = \sqrt{x}$

$(u^2-1)^2 = x = u^4-2u^2+1$

$dx = (4u^3-4u)\, du$

$\rightarrow \int \frac{u\,(4u^3-4u)\, du}{u^4-2u^2+1}$