MTH 174, Final Exam, Spring 2018 Name KEY

Instructions: Show all work. Give exact answers unless specifically asked to round. All complex numbers should be stated in standard form, and all complex fractions should be simplified. If you do not show work, problems will be graded as "all or nothing" for the answer only; partial credit will not be possible and any credit awarded for the work will not be available.

1. Find the area of the surface of revolution bounded by $y = 9 - x^2$ on $[0,2]$ revolved around the y-axis. (10 points)

$SA = 2\pi \int_0^2 x\sqrt{1+4x^2}\,dx$

$u = 1+4x^2$

$du = 8x\,dx$

$\frac{1}{8}du = x\,du$

$\frac{1}{8}\int u^{1/2}\,du =$

$y' = -2x$

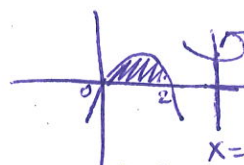
$\sqrt{1+[f'(x)]^2} = \sqrt{1+4x^2}$

$= \cancel{2}\pi\left[\frac{1}{\cancel{8}_{4_2}} \cdot \frac{2}{3}(1+4x^2)^{3/2}\right|_0^2$

$\frac{\pi}{6}(17^{3/2} - 1) \approx 36.18$

2. Find the volume of revolution bounded by $y = 2x - x^2, y = 0$, around the $x = 4$. (10 points)

$V = 2\pi \int_0^2 (4-x)(2x-x^2)\,dx$



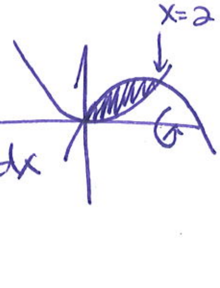
$x=4$

$= 2\pi \int_0^2 8x - 4x^2 - 2x^2 + x^3\,dx = 2\pi \int_0^2 8x - 6x^2 + x^3\,dx$

$= 2\pi\left[4x^2 - 2x^3 + \frac{1}{4}x^4\right|_0^2 = 2\pi[16 - 16 + 4] = 8\pi$

3. Find the volume of revolution bounded by $y = x^2, y = 4x - x^2$ around the x-axis. (10 points)

$x=2$



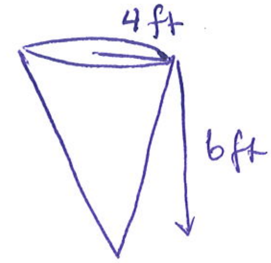
$V = \pi \int_0^2 (4x-x^2)^2 - (x^2)^2\,dx =$

$\pi \int_0^2 16x^2 - 8x^3 + \cancel{x^4} - \cancel{x^4}\,dx = \pi \int_0^2 16x^2 - 8x^3\,dx$

$= \pi\left[\frac{16}{3}x^3 - 2x^4\right|_0^2 = \frac{32\pi}{3}$

water density = k = 62.4

4. Find the work done in emptying a conical tank by pumping the water over the top edge. The tank is 8 ft. across and 6 ft. high. The tank is initially full. (10 points)

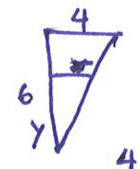


$$W = \int_0^6 K \cdot \pi \left(\tfrac{2}{3}y\right)^2 (6-y)\,dy$$

$$= \frac{4k\pi}{9} \int_0^6 6y^2 - y^3\,dy =$$

$$\frac{4k\pi}{9}\left[2y^3 - \tfrac{1}{4}y^4\right|_0^6 = \frac{4k\pi}{9}(108) = 48k\pi$$

$$\approx 2995.2\pi$$

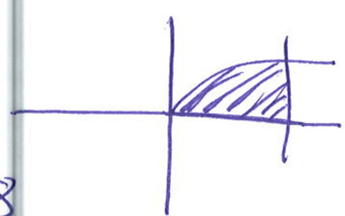


$\frac{4}{6} = \frac{x}{y}$

$\frac{2}{3}y = x = r$

5. Find the center of mass for the planar region bounded by $y = \sqrt{x}, y = 0, x = 4$. (12 points)

$$M = \int_0^4 \sqrt{x}\,dx = \tfrac{2}{3}x^{3/2}\Big|_0^4 = \frac{16}{3}$$



$$M_x = \int_0^4 (\sqrt{x})^2\,dx = \int_0^4 x\,dx = \tfrac{1}{2}x^2\Big|_0^4 = 8$$

$$M_y = \int_0^4 x\sqrt{x}\,dx = \int_0^4 x^{3/2}\,dx = \tfrac{2}{5}x^{5/2}\Big|_0^4 = \tfrac{2}{5}(32) = \frac{64}{5}$$

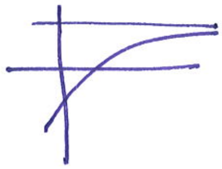
$$\bar{x} = \frac{\cancel{64}^{\,4}}{5} \cdot \frac{3}{\cancel{16}} = \frac{12}{5}$$

$$\bar{y} = \frac{\cancel{8}}{1} \cdot \frac{3}{\cancel{16}_{\,2}} = \frac{3}{2}$$

$$(\bar{x}, \bar{y}) = \left(\tfrac{12}{5}, \tfrac{3}{2}\right)$$

$$= (2.4, 1.5)$$

6. Determine if the **sequence** $a_n = \frac{n}{3n+2}$ is bounded and monotonic. (8 points)



for $n \geq 0$ never exceeds 3 bounded above.

bounded below by 0 when $n \geq 0$

$$3 = \frac{n}{3n+2}$$

$$9n + 6 = n$$

$8n = -6$ ← only 3 if negative

$$(a_n)' = \frac{1(3n+2)-(n)(3)}{(3n+2)^2} = \frac{\cancel{3n}+6-\cancel{3n}}{(3n+2)^2} = \frac{6}{(3n+2)^2} > 0$$

always increasing so montonic.

7. Find the sum of series. (6 points each)

a. $\sum_{n=0}^{\infty} 9\left(\frac{4}{7}\right)^n$

$$\frac{9}{1-\frac{4}{7}} = \frac{9}{\frac{3}{7}} = \overset{3}{\cancel{9}} \cdot \frac{7}{\cancel{3}} = 21$$

b. $\sum_{n=1}^{\infty} \frac{4}{n(n+2)}$

$$= \sum \frac{2}{n} - \frac{2}{n+2}$$

$$\frac{2}{1} + \frac{2}{2} - \left[\lim_{n \to \infty} \frac{2}{n+1} + \frac{2}{n+2}\right]$$

$$3 - 0 = 3$$

8. Determine if the series converges. Clearly state the test used. (5 points each) choose 3

a. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^5-1}}$

limit comparison w/ $\frac{1}{n^{5/2}}$ (converges by p-series)

$$\lim_{n \to \infty} \frac{\frac{1}{\sqrt{n^5-1}}}{\frac{1}{n^{5/2}}} = \lim_{n \to \infty} \frac{n^{5/2}}{\sqrt{n^5-1}} = 1$$

Converges

b. $\sum_{n=1}^{\infty} \frac{n+1}{n3^{n-1}}$

limit comp w/ $\frac{1}{3^n}$ (geometric converges)

$\lim_{n\to\infty} \frac{n+1}{n\, 3^{n-1}} \cdot \frac{3^n}{1} = 3$ converges

c. $\sum_{n=2}^{\infty} \frac{e^n}{3^n+1}$

root test

$\lim_{n\to\infty} \sqrt[n]{\frac{e^n}{3^n+1}} = \frac{e}{3} < 1$

converges

d. $\sum_{n=0}^{\infty} \frac{4^n}{(n+8)^n}$

root test

$\lim_{n\to\infty} \sqrt[n]{\frac{4^n}{(n+8)^n}} = \lim_{n\to\infty} \frac{4}{n+8} = 0 < 1$ converges

e. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}\sqrt{n}}{n+2}$

alternating

$\lim_{n\to\infty} \frac{\sqrt{n}}{n+2} = 0$ converges

f. $\sum_{n=0}^{\infty} \frac{n^2 e^n}{n!}$

ratio test

$\lim_{n\to\infty} \left| \frac{\cancel{(n+1)^2} e^{n+1}}{(n+1)\cancel{!}} \cdot \frac{\cancel{n!}}{\cancel{n^2} \cancel{e^n}} \right| = \lim_{n\to\infty} \frac{e}{n+1} = 0 < 1$ converges

lim = 1

9. Determine the interval and radius of convergence of x for the series below. Be sure to check the endpoints. (7 points each)

a. $\sum_{n=1}^{\infty} \frac{(-1)^n (x-4)^n}{n}$

$r = (x-4) < 1$ $\quad R = 1$

$-1 < x-4 < 1$
$+4 \quad +4 \quad +4$
$3 < x < 5$

$x = 3$
$\sum_{n=1}^{\infty} \frac{(-1)^n (-1)^n}{n}$ diverges $= \frac{1}{n}$

$x = 5$ $\sum_{n=1}^{\infty} \frac{(-1)^n (1)^n}{n}$ converges $\frac{(-1)^n}{n}$

$(3, 5]$ converges on

b. $\sum_{n=0}^{\infty} n! \left(\frac{x}{9}\right)^n$

$\lim_{n\to\infty} \left| \frac{(n+1)!}{\cancel{n!}} \right| \left(\frac{x}{9} \right) = \infty$

$R = 0$

$\{0\}$ converges on

10. Write the function $f(x) = \frac{2x}{(1-x)^2}$ as a power series centered at $c = 0$. (8 points)

$\frac{a}{1-r} = \sum_{n=0}^{\infty} ar^n$

$\frac{a}{(1-r)^2} = \sum_{n=1}^{\infty} anr^{n-1}$

$f(x) = 2x \sum_{n=1}^{\infty} x^{n-1} \cdot n$

$= \sum_{x=1}^{\infty} 2n x^n$

11. Find a Taylor polynomial for the function $f(x) = x\sin\frac{x}{3}$ centered at $c = 0$ for 4 terms. (10 points)

$$x\sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{x}{3}\right)^{2n+1}}{(2n+1)!} = x\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{3^{2n+1}\cdot(2n+1)!} =$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{3^{2n+1}(2n+1)!}$$

$$\approx \frac{(1)x^2}{3(1)} + \frac{(-1)x^4}{27\cdot 6} + \frac{(1)x^6}{3^5 5!} + \frac{(-1)x^8}{3^7\cdot 7!}$$

$$= \frac{x^2}{3} - \frac{x^4}{162} + \frac{x^6}{29160} - \frac{x^8}{11{,}022{,}480}$$

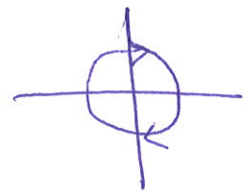
12. Use Taylor and power series (see the table at the end of the test) to find a power series for expression $f(x) = \int \frac{\ln(x+1)}{x}$. (8 points)

$$\int \frac{1}{x}\sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}\,dx = \sum_{n=1}^{\infty}\int \frac{(-1)^{n-1} x^{n-1}}{n}\,dx =$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n^2}$$

13. Sketch the parametric curve $x = \sin t\,, y = \cos t$. Label the orientation. Convert the equations back to rectangular coordinates. (8 points)

$x^2 + y^2 = \sin^2 t + \cos^2 t = 1$



$x^2 + y^2 = 1$

14. Find a parametric equation that passes through points $(4,1), and\ (5,-2)$. (8 points)

$\Delta x = 5 - 4 = 1$

$\Delta y = -2 - 1 = -3$

$r(t) = (t+4)\hat{i} + (1-3t)\hat{j}$

or $\begin{cases} x = t+4 \\ y = 1-3t \end{cases}$

15. Find the equation of the tangent line for the parametric curve given by $x = 3t - t^2, y = 2t^{3/2}$ at $t = \frac{1}{4}$. (8 points)

$$\frac{\frac{dy}{dt} = }{\frac{dx}{dt} = } \quad \frac{\cancel{2} \cdot \frac{3}{\cancel{2}} t^{1/2}}{3-2t} = \frac{3\left(\frac{1}{4}\right)^{1/2}}{3-2\left(\frac{1}{4}\right)} = \frac{3\left(\frac{1}{2}\right)}{3-\frac{1}{2}} = \frac{3}{5}$$

$x\left(\frac{1}{4}\right) = 3\left(\frac{1}{4}\right) - \left(\frac{1}{4}\right)^2 = \frac{11}{16}$

$y\left(\frac{1}{4}\right) = 2\left(\frac{1}{4}\right)^{3/2} = 2\left(\frac{1}{8}\right) = \frac{1}{4}$

$y - \frac{1}{4} = \frac{3}{5}\left(x - \frac{11}{16}\right)$

16. Find the area of one petal of $r = 6\cos 2\theta$. (12 points)

$6\cos 2\theta = 0$

$\cos 2\theta = 0$

$2\theta = \frac{\pi}{2}, -\frac{\pi}{2}$

$\theta = \frac{\pi}{4}, -\frac{\pi}{4}$

$\frac{1}{2}\int_{-\pi/4}^{\pi/4} (6\cos 2\theta)^2 \, d\theta =$

$18 \cdot 2\int_0^{\pi/4} \cos^2 2\theta \, d\theta =$

$\frac{18 \cdot \cancel{2}}{\cancel{2}} \int_0^{\pi/4} 1 + \cos 4\theta \, d\theta =$

$18\left[\theta + \frac{1}{4}\overset{0}{\sin 4\theta}\right|_0^{\pi/4} =$

$18\left(\frac{\pi}{4}\right) = \boxed{\frac{9\pi}{2}}$

17. Use $\vec{u} = \langle 3,2 \rangle$, $\vec{v} = \langle 1,-5 \rangle$ to find the following. (5 points each)

a. $\vec{u} + \vec{v}$

$\langle 4, -3 \rangle$

b. $||\vec{u}||$

$\sqrt{9+4} = \sqrt{13}$

c. Write $\vec{v}$ in polar form.

$\sqrt{1+25} = \sqrt{26}$

$\tan^{-1}\left(\frac{-5}{1}\right) = -1.3734$ radians or $-78.7°$

$\sqrt{26}\cos(-1.3734)\hat{i} + \sqrt{26}\sin(-1.3734)\hat{j}$

d. Write a unit vector in the direction of $\vec{u}$

$$\left\langle \frac{3}{\sqrt{13}}, \frac{2}{\sqrt{13}} \right\rangle$$

e. Find $\vec{u} \cdot \vec{v}$

$$3 - 10 = \boxed{-7}$$

18. Approximate $\int_0^4 x^2 e^{x}\, dx$ using (12 points) Simpson's Rule with $n = 4$

$x_0 = 0,\ x_1 = 1,\ x_2 = 2,\ x_3 = 3,\ x_4 = 4$

$$\frac{4-0}{3(4)}\left[\cancel{0e^0} + 4(1)^2e^1 + 2(2)^2e^2 + 4(3)^2e^3 + 4^2e^4\right]$$

$$\frac{1}{3}\left[4e + 8e^2 + 36e^3 + 16e^4\right] \approx 555.545$$

19. Find the arc length for $x = \frac{1}{3}(y^2+4)^{3/2}$ on $[0,4]$. Set up the integral. You may evaluate it numerically in your calculator. (6 points)

$$x' = \frac{1}{\cancel{3}}(y^2+4)^{1/2}\cdot\frac{\cancel{3}}{\cancel{2}}\cdot\cancel{2}y = y(y^2+4)^{1/2}$$

$$\int_0^4 \sqrt{1+y^2(y^2+4)}\,dy \approx 27.854$$

20. Integrate. (7 points each)

a. $\int \frac{\cos\left(\frac{1}{\theta}\right)}{\theta^2}\,d\theta$

$u = \frac{1}{\theta}$

$du = \frac{-1}{\theta^2}$

$$\int -\cos u\,du = -\sin u + C$$

$$-\sin\left(\frac{1}{\theta}\right) + C$$

b. $\int \arctan x\,dx$

$u = \arctan x$ $\quad dv = dx$

$du = \frac{1}{1+x^2}dx$ $\quad v = x$

$$x\arctan x - \int \frac{x}{1+x^2}\,dx =$$

$$x\arctan x - \frac{1}{2}\ln|1+x^2| + C$$

c. $\int \frac{1}{(4-t^2)^{3/2}} dt$

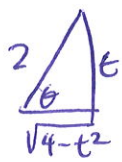
$t = 2\sin\theta$

$dt = 2\cos\theta \, d\theta$

$\sqrt{4-t^2} = 2\cos\theta$

$$\int \frac{2\cos\theta \, d\theta}{(2\cos\theta)^3} = \int \frac{\cancel{2}\cancel{\cos\theta}}{\underset{4}{\cancel{8}}\cos^{\cancel{3}2}\theta} d\theta = \frac{1}{4}\int \sec^2\theta \, d\theta$$

$$= \frac{1}{4}\tan\theta + C$$



$$= \frac{1}{4}\frac{t}{\sqrt{4-t^2}} + C$$

d. $\int \frac{9}{4x^2-1} dx$

$$\frac{A}{2x+1} + \frac{B}{2x-1} = \frac{2Ax - A + 2Bx + B}{4x^2-1} = \frac{9}{4x^2-1}$$

$$\int \frac{-\frac{9}{2}}{2x+1} + \frac{\frac{9}{2}}{2x-1} dx$$

$2A + 2B = 0 \rightarrow A + B = 0$

$-A + B = 9 \qquad -A + B = 9$

$2B = 9$

$B = \frac{9}{2}$

$A = -\frac{9}{2}$

$$-\frac{9}{4}\ln|2x+1| + \frac{9}{4}\ln|2x-1| + C$$

21. Rewrite $\int \sec^8 x \, dx$ so that it can be integrated with substitution. You do not need to integrate. (6 points)

$$\int (\sec^2 x)^3 \sec^2 x \, dx$$

$$\int (1+\tan^2 x)^3 \sec^2 x \, dx$$

22. Rewrite $\int \frac{x^3+5x^2-x}{(x^2-1)(x+3)^3(x^2+1)}\,dx$ with partial fractions. Do not solve for the constants or integrate. (6 points)

$$X^2-1=(X-1)(X+1)$$

$$\frac{A}{X-1}+\frac{B}{X+1}+\frac{C}{X+3}+\frac{D}{(X+3)^2}+\frac{E}{(X+3)^3}+\frac{FX+G}{X^2+1}$$

23. Evaluate the limit $\lim_{x\to 0}\frac{\ln x}{\cot x}$. (6 points)

$-\frac{\infty}{\infty}$

$$\lim_{X\to 0}\frac{\frac{1}{X}}{-\csc^2 X}=\lim_{X\to 0}-\frac{1}{X}\cdot \sin^2 X \overset{0/0}{=} \lim_{X\to 0}\frac{-2\sin X\cos X}{1}=0$$

24. Evaluate the improper integral $\int_1^\infty \frac{1}{x\ln x}\,dx$ or show that it diverges. (8 points)

$u=\ln x$ $\quad X=\infty \to u=\infty$, $\quad X=1\to u=0$

$du=\frac{1}{X}\,dx$

$$\int_0^\infty \frac{1}{u}\,du=\ln u\Big|_0^\infty$$

diverges

since both ends go to ∞

Useful formulas:

$$\frac{dy}{dx} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \dots \qquad R = 1$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \qquad R = \infty$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \qquad R = \infty$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \qquad R = \infty$$

$$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots \qquad R = 1$$

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots \qquad R = 1$$

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \dots \qquad R = 1$$