MTH 174, Exam #1, Part 1, Spring 2018 Name KEY

Instructions: Show all work. Give exact answers unless specifically asked to round. All complex numbers should be stated in standard form, and all complex fractions should be simplified. If you do not show work, problems will be graded as "all or nothing" for the answer only; partial credit will not be possible and any credit awarded for the work will not be available.

1. Integrate the following integrals using the Table of Integrals provided. Substitution may be needed. Report the formula used. (7 points each)
   a. $\int t^3 \cos t \, dt$

$\int x^n \cos x \, dx = x^n \sin x - n\int x^{n-1} \sin x \, dx$ 4.16

$\int x^n \sin x \, dx = -x^n \cos x - n\int x^{n-1} \cos x \, dx$ 4.8

$t^3 \sin t - 3\int x^2 \sin x \, dx$

$t^3 \sin t - 3[-t^2 \cos t - 2\int x \cos x \, dx]$

$t^3 \sin t + 3t^2 \cos t + 6\int t \cos t \, dt$

$t^3 \sin t + 3t^2 \cos t + 6t \sin t - 6\int \sin t \, dt$

$\boxed{t^3 \sin t + 3t^2 \cos t + 6t \sin t - 6 \cos t + C}$

   b. $\int x^2\sqrt{2 + 9x^2}dx$ $= 3\int x^2 \sqrt{\frac{2}{9} + x^2}$

$\int x^2 \sqrt{x^2 \pm a^2} \, dx = \frac{x}{8}(2x^2 \pm a^2)\sqrt{x^2 \pm a^2} - \frac{a^4}{8} \ln\left|x + \sqrt{x^2 \pm a^2}\right| + C$

$3\left[\frac{x}{8}(2x^2 + \frac{2}{9})\sqrt{x^2 + \frac{2}{9}} - \frac{4/81}{8} \ln\left|x + \sqrt{x^2 + \frac{2}{9}}\right| + C\right]$ $a = \frac{\sqrt{2}}{3}$

3.29

$\boxed{\frac{3x}{8}(2x^2 + \frac{2}{9})\sqrt{x^2 + \frac{2}{9}} - \frac{1}{54} \ln\left|x + \sqrt{x^2 + \frac{2}{9}}\right| + C}$

c. $\int \frac{e^x}{1-\tan e^x}\,dx$ $\quad u = e^x$ $\quad du = e^x dx$ $\quad \int \frac{1}{1-\tan u}\,du$

$\int \frac{1}{1 \pm \tan x}\,dx = \frac{1}{2}\left[x \pm \ln|\cos x \pm \sin x|\right] + C$ $\quad$ 4.37

$\frac{1}{2}\left[u - \ln|\cos u - \sin u|\right] + C$

$$\boxed{\frac{1}{2}e^x - \frac{1}{2}\ln|\cos e^x - \sin e^x| + C}$$

2. Approximate $\int_1^4 \sqrt{x}e^x\,dx$ using (6 points each) $\quad \Delta x = \frac{4-1}{6} = \frac{3}{6} = \frac{1}{2}$
   a. Trapezoidal Rule with $n = 6$

$\frac{1}{2}\cdot\frac{1}{2}\left[e^1 + 2\sqrt{1.5}\,e^{1.5} + 2\sqrt{2}\,e^2 + 2\sqrt{2.5}\,e^{2.5} + 2\sqrt{3}\,e^3 + 2\sqrt{3.5}\,e^{3.5} + 2e^4\right]$

≈ 94.62989

   b. Simpson's Rule with $n = 4$ $\quad \Delta x = \frac{4-1}{4} = \frac{3}{4}$

$\frac{1}{4}\left[e^1 + 4e^{1.75}\sqrt{1.75} + 2\sqrt{2.5}\,e^{2.5} + 4\sqrt{3.25}\,e^{3.25} + 2e^4\right]$

≈ 91.71657

c. Find n for Trapezoidal Rule needed to estimate the integral to $E \leq 0.0001$.

$$10^{-4} = \frac{134.8(4-1)^3}{12n^2}$$

$$n^2 = \frac{134.8 \cdot 27}{12} 10^4$$

$$n \approx 1741.55$$

$$\boxed{n = 1742}$$

$x^{1/2}$

$\frac{1}{2}x^{-1/2}$

$-\frac{1}{4}x^{-3/2}$

$\frac{3}{8}x^{-5/2}$

$-\frac{15}{16}x^{-7/2}$

$f = x^{1/2}e^x$

$f' = \frac{1}{2}x^{-1/2}e^x + x^{1/2}e^x = e^x\left(\frac{1}{2}x^{-1/2} + x^{1/2}\right)$

$f'' = e^x\left(\frac{1}{2}x^{-1/2} + x^{1/2}\right) + e^x\left(-\frac{1}{4}x^{-3/2} + \frac{1}{2}x^{-1/2}\right)$

$= e^x\left(-\frac{1}{4}x^{-3/2} + x^{-1/2} + x^{1/2}\right)$

$f''' = e^x\left(\frac{3}{8}x^{-5/2} - \frac{3}{4}x^{-3/2} - \frac{3}{2}x^{-1/2} + x^{1/2}\right)$

$f^{IV} = e^x\left(-\frac{15}{16}x^{-7/2} + \frac{12}{8}x^{-5/2} - \frac{6}{4}x^{-3/2} + \frac{4}{2}x^{-1/2} + x^{1/2}\right)$

max $|f''(x)|$ at 4 $= 134.8$ (approx)

d. Find n for Simpson's Rule needed to estimate the integral to $E \leq 0.0001$.

max $|f^{IV}(x)|$ at 4 $= 155.7$

$$10^{-4} = \frac{(4-1)^5\ 155.7}{180n^4}$$

$$n^4 = \frac{(4-1)^5\ 155.7}{180} 10^4$$

$$n^4 = \frac{243 \cdot 155.7}{180} 10^4$$

$$n \approx 38.076$$

$$\boxed{n = 40}$$

Formulas:

$$E_T \leq \frac{Max|f''(x)|(b-a)^3}{12n^2}$$

$$E_M \leq \frac{Max|f^{IV}(x)|(b-a)^5}{180n^4}$$

MTH 174, Exam #1, Part 2, Spring 2018 Name KEY

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1. Integrate. (6 points each)

a. $\int \frac{\cos\left(\frac{1}{\theta}\right)}{\theta^2} d\theta$

$u = \frac{1}{\theta}$

$-du = +\frac{1}{\theta^2} d\theta$

$\int -\cos u \, du = -\sin u + C$

$= \boxed{-\sin\left(\frac{1}{\theta}\right) + C}$

b. $\int \frac{2x+1}{\sqrt{x+4}} dx$

$u = \sqrt{x+4}$

$u^2 = x+4$

$u^2 - 4 = x$

$2u\,du = dx$

$2(u^2-4)+1 = 2x+1$

$2u^2-8+1 = 2u^2-7$

$\int \frac{2u^2-7}{\cancel{u}} 2\cancel{u}\,du = \int 4u^2 - 14 \, du$

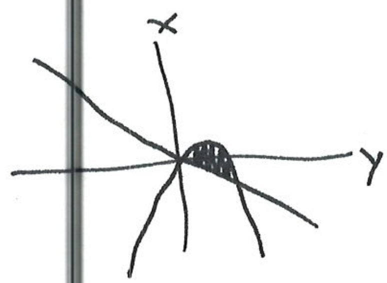
$\frac{4}{3}u^3 - 14u + C$

$\boxed{\frac{4}{3}(x+4)^{3/2} - 14\sqrt{x+4} + C}$

2. Set up an integral to find the area between the pair of curves. You do not need to integrate. Sketch the regions. (5 points each)
   a. $f(y) = y(2-y), g(y) = -y$

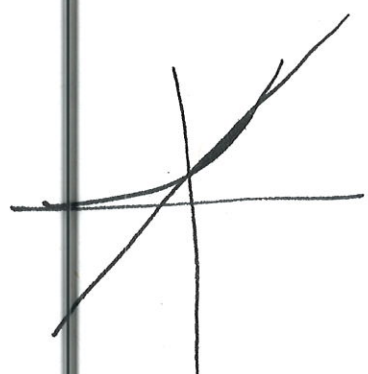
intersections : 0, 3

$\int_0^3 (y(2-y)+y)\,dy$



   b. $f(x) = 2^x, g(x) = \frac{3}{2}x + 1$

$\int_0^2 \frac{3}{2}x+1-2^x\,dx$



intersections: 0, 2

3. Find the arc length for $x = \frac{1}{3}(y^2+2)^{3/2}$ on $[0,4]$. Set up the integral. You may evaluate it numerically in your calculator. (5 points)

$$x' = \frac{1}{3}\cdot\frac{3}{2}(y^2+2)^{1/2}\cdot 2y = y\sqrt{y^2+2}$$

$$\int_0^4 \sqrt{1+y^2(y^2+2)}\,dy = \int_0^4 \sqrt{y^4+2y^2+1}\,dy =$$

$$\int_0^4 \sqrt{(y^2+1)^2}\,dy = \int_0^4 y^2+1\,dy = \frac{1}{3}y^3+y\Big]_0^4 =$$

$$\frac{1}{3}\cdot 64+4 = \boxed{76/3}$$

4. Integrate. (6 points each)

a. $\int \arctan x\,dx$

$u = \arctan x \qquad dv = dx$

$du = \frac{1}{1+x^2}dx \qquad v = x$

$$x\arctan x - \int \frac{x}{1+x^2}dx$$

$$\boxed{x\arctan x - \frac{1}{2}\ln|1+x^2| + C}$$

b. $\int e^{4x}\cos 2x\,dx$

$u = \cos 2x \qquad dv = e^{4x}dx$

$du = -2\sin 2x\,dx \qquad v = \frac{1}{4}e^{4x}$

$-\frac{2}{4}e^{4x}\sin 2x -$

$$\frac{1}{4}e^{4x}\cos 2x + \frac{1}{2}\int e^{4x}\sin 2x\,dx$$

$u = \sin 2x \qquad dv = e^{4x}dx$

$du = 2\cos 2x\,dx \qquad v = \frac{1}{4}e^{4x}$

$$\frac{1}{4}e^{4x}\cos 2x + \frac{1}{8}e^{4x}\sin 2x - \frac{1}{4}\int e^{4x}\cos 2x\,dx = \int e^{4x}\cos 2x\,dx$$

$$+\frac{1}{4}\int e^{4x}\cos 2x\,dx \qquad \frac{5}{4}\int e^{4x}\cos 2x\,dx$$

$$\frac{4}{5}\cdot\left[\frac{1}{4}e^{4x}\cos 2x + \frac{1}{8}e^{4x}\sin 2x = \frac{5}{4}\int e^{4x}\cos 2x\,dx\right]\cdot\frac{4}{5}$$

$$\boxed{\frac{1}{5}e^{4x}\cos 2x + \frac{1}{10}e^{4x}\sin 2x + C}$$

5. Rewrite $\int \sin^8 x\, dx$ so that it can be integrated with substitution. You do not need to integrate. (5 points)

$$(\sin^2 x)^4$$

$$\frac{1}{16}\int (1 - \cos 2x)^4\, dx = \frac{1}{16}\int 1 - 4\cos 2x + 6\cos^2 2x - 4\cos^3 2x + \cos^4 2x\, dx$$

$$6\tfrac{1}{2}(1+\cos 4x) \qquad \tfrac{1}{4}(1+\cos 4x)^2$$

$$3 + 3\cos 4x \qquad \tfrac{1}{4}(1 + 2\cos 4x + \cos^2 4x)$$

$$\tfrac{1}{2}(1+\cos 8x)$$

$$\frac{1}{16}\int 1 - 4\cos 2x + 3 + 3\cos 4x - 4(1-\sin^2 2x)\cos 2x + \tfrac{1}{4} + \tfrac{1}{2}\cos 4x + \tfrac{1}{8} + \tfrac{1}{8}\cos 8x\, dx$$

$$= \frac{1}{16}\int \frac{35}{8} - 4\cos 2x + \frac{7}{2}\cos 4x + \frac{1}{8}\cos 8x - 4(1-\sin^2 2x)\cos 2x\, dx$$

$u = \sin 2x$

$$4\int (1-u^2)\,\tfrac{1}{2}\,du$$

6. Use trig substitution to integrate $\int \frac{1}{(1-t^2)^{5/2}}\,dt$. (8 points)

$t = \sin\theta$

$1 - t^2 = 1 - \sin^2\theta$

$= \cos^2\theta$

$dt = \cos\theta$

$$\int \frac{\cos\theta\, d\theta}{(\cos^2\theta)^{5/2}} = \int \frac{\cos\theta\, d\theta}{\cos^5\theta} = \int \frac{1}{\cos^4\theta}\, d\theta$$

$$= \int \sec^4\theta\, d\theta = \int (1+\tan^2\theta)\sec^2\theta\, d\theta$$

$u = \tan\theta$

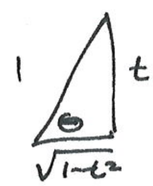
$du = \sec^2\theta$

$$\int (1+u^2)\, du =$$

$$u + \frac{1}{3}u^3 + C$$

$$\tan\theta + \frac{1}{3}\tan^3\theta + C$$

$$\boxed{\frac{t}{\sqrt{1-t^2}} + \frac{t^3}{3(1-t^2)^{3/2}} + C}$$



7. Rewrite $\int \frac{x^4+3x^2+x}{(x-1)(x+2)^2(x^2+4)}\,dx$ with partial fractions. Do not solve for the constants or integrate. (5 points)

$$\int \frac{A}{x-1} + \frac{B}{x+2} + \frac{C}{(x+2)^2} + \frac{Dx+E}{x^2+4}\, dx$$

8. Integrate $\int \frac{2}{9x^2-1} dx$ with partial fractions. (8 points)

$(3x+1)(3x-1)$

$$\frac{A}{3x+1} + \frac{B}{3x-1} = \frac{3Ax - A + 3Bx + B}{(3x+1)(3x-1)} \Rightarrow \begin{matrix} 3A+3B=0 \\ -A+B=2 \end{matrix} \quad \begin{matrix} A=-1 \\ B=1 \end{matrix}$$

$$\int \frac{-1}{3x+1} + \frac{1}{3x-1}\, dx = \boxed{-\frac{1}{3}\ln|3x+1| + \frac{1}{3}\ln|3x-1| + C}$$

9. Evaluate the limits. (4 points each)

a. $\lim_{x \to 1} \frac{\ln x}{\sin \pi x}$

$\frac{0}{0}$ $\quad \lim_{x\to 1} \frac{\frac{1}{x}}{\pi \cos \pi x} = \frac{1}{\pi(-1)} = \boxed{\frac{-1}{\pi}}$

b. $\lim_{x \to 1} (\ln x)^{x-1}$

0^0 $\quad \frac{\infty}{\infty}$

$$e^{\lim_{x\to 1} \ln (\ln x)^{x-1}} = e^{\lim_{x\to 1} (x-1)\ln(\ln x)} = e^{\lim_{x\to 1} \frac{\ln(\ln x)}{\frac{1}{x-1}}}$$

$$e^{\lim_{x\to 1} \frac{\frac{1}{x \ln x}}{-\frac{1}{(x-1)^2}}} = e^{\lim_{x\to 1} -\frac{(x-1)^2}{x \ln x}} = e^{\lim_{x\to 1} \frac{-2(x-1)}{\ln x + 1}} = e^{\frac{-2(0)}{0+1}} =$$

$$e^0 = \boxed{1}$$

10. Evaluate the improper integrals or show that it diverges. (7 points each)

a. $\int_0^\infty \frac{1}{e^x+e^{-x}} dx$ $\frac{e^x}{e^x}$

$u = e^x$

$du = e^x dx$

$\int_0^\infty \frac{e^x}{e^{2x}+1} dx$

$\int_1^\infty \frac{du}{u^2+1} = \arctan u \Big|_1^\infty$

$\lim_{u \to \infty} \arctan u - \pi/4 = \pi/2 - \pi/4 = \pi/4$

Converges

b. $\int_1^\infty \frac{1}{x \ln x} dx$ $= \ln(\ln x) \Big|_1^\infty$

$\lim_{x \to \infty} \ln(\ln x) - \lim_{x \to 1} \ln(\ln x)$

$\infty - (-\infty) = \infty$

diverges