MAT 212, Quiz #9, Spring 2017 Name KEY

Instructions: Show all work. Give exact answers unless specifically asked to round. Be sure to answer all parts of each question.

1. Find the third derivative of $y = \sum_{n=0}^{\infty} a_n x^n$ and reindex it so that the resulting series starts at $n = 0$.

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

$$y''' = \sum_{n=3}^{\infty} n(n-1)(n-2) a_n x^{n-3}$$

$$= \sum_{n=0}^{\infty} (n+3)(n+2)(n+1) a_{n+3} x^n$$

2. Determine if $f(x) = e^x$ and $g(x) = xe^{-x}$ are orthogonal under the inner product $\langle f, g \rangle = \int_{-1}^{1} f(x)g(x)dx$.

$$\int_{-1}^{1} e^x (xe^{-x})dx = \int_{-1}^{1} x\,dx = \frac{1}{2}x^2\Big|_{-1}^{1} = \frac{1}{2}(1)^2 - \frac{1}{2}(-1)^2 = 0$$

yes, they are orthogonal

3. Solve the equation $(2x^2 - 1)y' + 2y = 0$ using series methods.

$$y = \sum_{n=0}^{\infty} a_n x^n \quad y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$2x^2 \sum_{n=1}^{\infty} n a_n x^{n-1} - \sum_{n=1}^{\infty} n a_n x^{n-1} + 2\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=1}^{\infty} 2n a_n x^{n+1} - \sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} 2a_n x^n = 0$$

$$\sum_{n=2}^{\infty} 2(n-1) a_{n-1} x^n - \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n + \sum_{n=0}^{\infty} 2a_n x^n = 0$$

$$-a_1 - 2a_2 x + 2a_0 + 2a_1 x + \sum_{n=2}^{\infty} 2(n-1) a_{n-1} x^n - \sum_{n=2}^{\infty} (n+1) a_{n+1} x^n + \sum_{n=2}^{\infty} 2a_n x^n = 0$$

$2a_0 = a_1$ $\quad \cancel{2}a_1 = \cancel{2}a_2 = 2a_0$

$$\sum_{n=2}^{\infty} \left[2(n-1) a_{n-1} - (n+1) a_{n+1} + 2a_n\right] x^n = 0$$

3(cont'd)

$a_1 = 2a_0$

$a_2 = 2a_0$

$2(n-1)a_{n-1} - (n+1)a_{n+1} + 2a_n = 0$

$$\frac{2(n-1)a_{n-1} + 2a_n}{n+1} = \cancel{(n+1)}a_{n+1}$$

$a_1 = 2a_0$

$a_2 = 2a_0$

$a_3 = \frac{8}{3}a_0$

$a_4 = \frac{10}{3}a_0$

$a_5 = \frac{68}{15}a_0$

$n=2$

$$\frac{2(1)}{3}a_1 + \frac{2}{3}a_2 = a_3$$

$$\frac{2}{3}(2a_0) + \frac{2}{3}(2a_0) = a_3$$

$$\frac{4}{3}a_0 + \frac{4}{3}a_0 = \frac{8}{3}a_0 = a_3$$

$n=3$

$$\frac{2(2)}{4}a_2 + \frac{\cancel{2}}{\cancel{4}}\left(\frac{\overset{2}{\cancel{8}}}{3}a_0\right) = a_4$$

$$(1)(2a_0) + \overset{a_3}{\frac{4}{3}a_0} = \frac{10}{3}a_0 = a_4$$

$n=4$

$$\frac{2(3)}{5}a_3 + \frac{2}{5}a_4 =$$

$$\frac{6}{5}\left(\frac{8}{3}a_0\right) + \frac{2}{5}\left(\frac{10}{3}\right)a_0 = \frac{68}{15}a_0 = a_5$$

$$y(x) = a_0\left[1 + 2x + 2x^2 + \frac{8}{3}x^3 + \frac{10}{3}x^4 + \frac{68}{15}x^5 + \dots\right]$$