MAT 212, Exam #2, Spring 2017 Name KEY

Instructions: Show all work. Answers without work required to obtain the solution will not receive full credit. Some questions may contain multiple parts: be sure to answer all of them. Give exact answers unless specifically asked to estimate.

1. Solve the second-order ODEs for the general solution. (9 points each)

   a. $2y'' - y' - y = 0$

   $2r^2 - r - 1 = 0$

   $(2r+1)(r-1) = 0$

   $r = -\frac{1}{2}, r = 1$

   $y = c_1 e^{-1/2 t} + c_2 e^{t}$

   b. $y'' - 2y' + 2y = 0$

   $r^2 - 2r + 2 = 0$

   $r = \dfrac{2 \pm \sqrt{4-8}}{2} = \dfrac{2 \pm 2i}{2} = 1 \pm i$

   $y = c_1 e^{t} \cos t + c_2 e^{t} \sin t$

   c. $x^2 y'' + 2xy' - 6y = 0$

   $n(n-1) + 2n - 6 = 0$

   $n^2 - n + 2n - 6 = 0$

   $n^2 + n - 6 = 0$

   $(n+3)(n-2) = 0$

   $n = -3, n = 2$

   $y = c_1 t^{-3} + c_2 t^{2}$

d. $y'' - 18y' + 81y = 0$

$r^2 - 18r + 81 = 0$

$(r-9)^2 = 0$

$r = 9$ repeated

$y = c_1 e^{9t} + c_2 t e^{9t}$

e. $y''' + 2y'' - y' - 2y = 0$

$r^3 + 2r^2 - r - 2 = 0$

$r^2(r+2) - 1(r+2) = 0$

$(r^2-1)(r+2) = 0$

$(r+1)(r-1)(r+2) = 0$

$r = -1,\ r = 1,\ r = -2$

$y = c_1 e^{-t} + c_2 e^{t} + c_3 e^{-2t}$

2. The table below gives the solution to the second order constant coefficient homogeneous equation, and the forcing function $F(x)$ or $F(t)$. Determine the Ansatz for the method of undetermined coefficients in each case. (4 points each)

	y_1	y_2	y_3	$F(x)$ or $F(t)$	Ansatz
a.	e^{-2x}	e^{3x}	NA	$2\sin 3x$	$A\cos 3x + B\sin 3x$
b.	$e^{-x}\cos x$	$e^{-x}\sin x$	NA	$e^x \sin x$	$Ae^x\cos x + Be^x \sin x$
c.	e^x	$e^{-x/2}\cos\left(\frac{\sqrt{3}}{2}x\right)$	$e^{-x/2}\sin\left(\frac{\sqrt{3}}{2}x\right)$	$e^x + 7$	$Axe^x + B$
d.	t	1	e^{-t}	$t + e^{-t}$	$At^3 + Bt^2 + Cte^{-t}$
e.	$\sin t$	$\cos t$	e^{-t}	$\cos^2 t$	$A + B\cos 2t + C\sin 2t$

$= \frac{1}{2}(1 + \cos 2t)$

3. What is the difference between the natural frequency of the system, and a quasi-frequency? How is each obtained? (4 points)

The natural frequency is what the frequency is when there is no damping. The quasi-frequency depends on the damping. The natural frequency is $\omega = \sqrt{\frac{k}{m}}$. The quasi-frequency is $\frac{\sqrt{\gamma^2 - 4km}}{2m} = \omega$.

4. What conditions are needed in a forced oscillation system to achieve beats? (4 points)

The system needs to be undamped and the forcing to the system needs to be similar to, but not the same as, the natural frequency.

5. Use the method of reduction of order to solve $(1-x^2)y'' - 2xy' + 2y = 0$, given $y_1(x) = x$. (12 points)

$y_2 = v \cdot y_1$ $\quad y_1' = 1$

$y_2' = v'y_1 + vy_1'$ $\quad y_1'' = 0$

$y_2'' = v''y_1 + 2v'y_1' + \cancel{y_1''}$

$y_2 = vx$

$y_2' = v'x + v$

$y_2'' = v''x + 2v'$

$(1-x^2)(v''x + 2v') - 2x(v'x + v) + 2vx = 0$

$xv'' + 2v' - v''x^3 - 2v'x^2 - 2x^2v' - \cancel{2vx} + \cancel{2vx} = 0$

$v''(x - x^3) + 2v'(1 - 2x^2)$

$xv''(1-x^2) = -2v'(1-2x^2)$

$v'' = \frac{-2(1-2x^2)}{x(1-x^2)} v'$

let $u = v'$

$v'' = \frac{du}{dx}$

$\frac{A}{x} + \frac{B}{1-x} + \frac{C}{1+x} = \frac{-2+4x^2}{\sim}$

$A - Ax^2 + Bx + Bx^2 + Cx - Cx^2 = -2+4x^2$

$-Ax^2 + Bx^2 - Cx^2 = 4x^2$

$A = -2$

$Bx + Cx = 0x$ $\quad B + C = 0 \Rightarrow B = -C$

$-2 + B + B = 4$ $\quad 2B = 6$ $\quad B = 3,\ C = -3$

$v = \int \frac{1}{x^2(1-x)^3(1+x)^3}\, dx$

$\frac{du}{dx} = \frac{-2+4x^2}{x(1-x)(1+x)} u \Rightarrow \int \frac{du}{u} = \int \frac{-2+4x^2}{x(1-x)(1+x)}\, dx$

$y_2 = x \int \frac{1}{x^2(1-x)^3(1+x)^3}\, dx$

$= \int \frac{-2}{x} + \frac{3}{1-x} - \frac{3}{1+x}\, dx \Rightarrow \ln u = -2\ln x - 3\ln(1-x) - 3\ln(1+x)$

$u = x^{-2}(1-x)^{-3}(1+x)^{-3}$

6. Set up the differential equation to solve the spring-mass problem with a 12 lbs. weight that stretches a spring 6 in. and a dashpot that provides 3 lbs. of resistance for every ft/s of velocity. The weight is pulled from an additional one foot from equilibrium and then released from rest. [You don't need to solve.] Is the system undamped, underdamped, critically damped or overdamped? (8 points)

$12 = k(1/2)$ $\qquad \frac{12}{32} = \frac{3}{8} = m$

$k = 24$

underdamped

$$\frac{8}{3}\left(\frac{3}{8}y'' + 3y' + 24y = 0\right)$$

$$y'' + 8y' + 64y = 0$$

$$r^2 + 8r + 64 = 0$$

$$r = \frac{-8 \pm \sqrt{64 - 4(64)}}{2} =$$

$$r = -4 \pm 4\sqrt{3}i$$

7. Use the method of variation of parameters to find the particular solution to $y'' + 6y' + 9y = 4e^{2t} + e^{-t}$ [Hint: $Y(t) = -y_1 \int \frac{y_2 g}{W} dt + y_2 \int \frac{y_1 g}{W} dt$.] (10 points)

$$r^2 + 6r + 9 = 0$$

$$(r+3)^2 = 0 \quad r = -3 \text{ repeated} \qquad e^{-3t},\ te^{-3t}$$

$$W = \begin{vmatrix} e^{-3t} & te^{-3t} \\ -3e^{-3t} & e^{-3t} - 3te^{-3t} \end{vmatrix} = e^{-6t} \cancel{- 3te^{-6t}} \cancel{+ 3te^{-6t}} = e^{-6t}$$

$$Y(t) = -e^{-3t} \int \frac{te^{-3t}(4e^{2t} + e^{-t})}{e^{-6t}} dt + te^{-3t} \int \frac{e^{-3t}(4e^{2t} + e^{-t})}{e^{-6t}} dt$$

$$= -e^{-3t} \int \underbrace{te^{5t} + te^{2t}}_{t(e^{5t} + e^{2t})} dt + te^{-3t} \int e^{5t} + e^{2t} dt$$

$$= e^{-3t}\left[t\left(\tfrac{1}{5}e^{5t} + \tfrac{1}{2}e^{2t}\right) - \left(\tfrac{1}{25}e^{5t} + \tfrac{1}{4}e^{2t}\right)\right] + te^{-3t}\left(\tfrac{1}{5}e^{5t} + \tfrac{1}{2}e^{2t}\right)$$

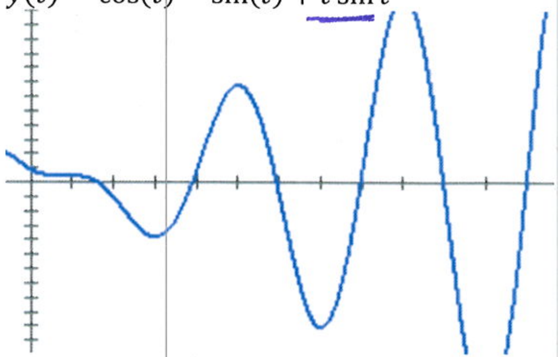
$$= \tfrac{1}{5}te^{2t} + \cancel{\tfrac{1}{2}te}e^{-t} - \tfrac{1}{25}e^{2t} - \tfrac{1}{4}e^{-t} + \tfrac{1}{5}te^{2t} + \tfrac{1}{2}te^{-t}$$

$$= \tfrac{2}{5}te^{2t} + te^{-t} - \tfrac{1}{25}e^{2t} - \tfrac{1}{4}e^{-t}$$

$\pm$	u	dv
+	t	$e^{5t} + e^{2t}$
−	1	$\frac{1}{5}e^{5t} + \frac{1}{2}e^{2t}$
+	0	$\frac{1}{25}e^{5t} + \frac{1}{4}e^{2t}$

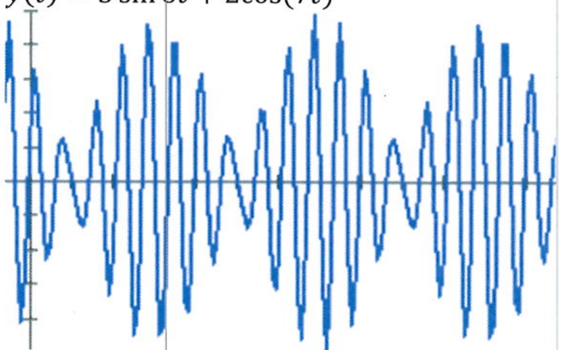
8. Below are the graphs of solutions to forced spring problems. Determine if the solution models resonance or beats (or neither). Explain your reasoning. (3 points each)

a. $y(t) = \cos(t) - \sin(t) + t \sin t$



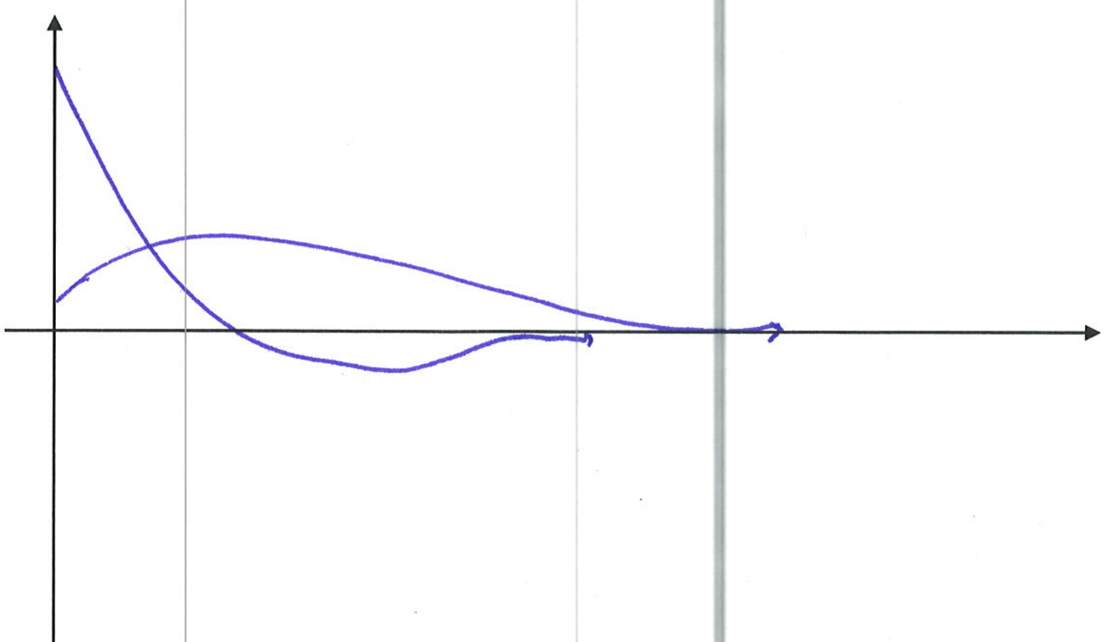
resonance
Same frequency
multiple of t makes frequency increase over time

b. $y(t) = 3\sin 6t + 2\cos(7t)$



beats
Similar frequencies but not the same

9. Sketch a graph of what an overdamped spring system looks like. (4 points)



10. For each of the solutions below to a forced oscillation system, state i) the transient or steady state solution, ii) whether the system is undamped, underdamped, critically damped or overdamped, and iii) if resonance or beats occurs. (5 points each)

a. $y(t) = \underbrace{e^{-t}(c_1 \cos 5t + c_2 \sin 5t)}_{\text{transient}} + \underbrace{5 \cos 4t + 4 \sin 4t}_{\text{steady state}}$

transient   steady state

underdamped

no resonance or beats

b. $y(t) = \underbrace{c_1 e^{-t} + c_2 e^{-2t}}_{\text{transient}} + \overline{\sin 3t}$

transient   steady state

overdamped   no resonance or beats

c. $y(t) = \underbrace{c_1 \cos 2t + c_2 \sin 2t}_{} + \frac{1}{6} t \cos 2t$

← all steady state

undamped   exhibits resonance

11. Use Abel's Theorem to find the value of the Wronskian for $y'' + 2xy' + 8y = 0$. (4 points)

$W = e^{-\int 2x\,dx} = e^{-x^2}$