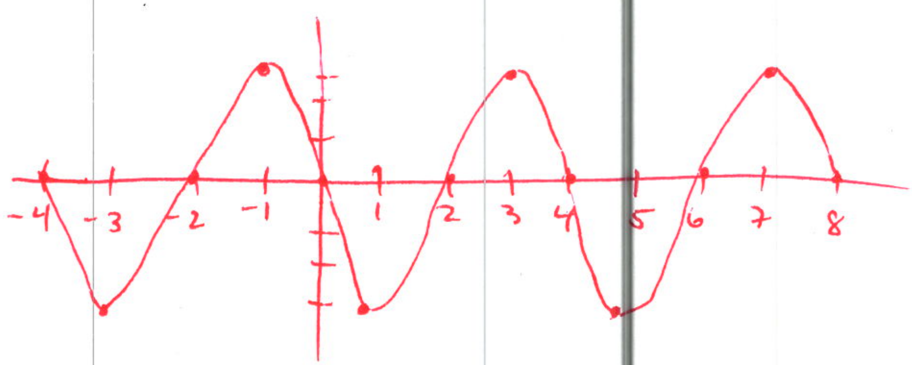
MAT 142, Final Exam, Spring 2017 Name KEY

Instructions: Show all work. Give exact answers unless specifically asked to round. Complete all parts of each question. Questions that provide only answers and no work will not receive full credit. If you use your calculator (only when problems don't instruct you to do the problem by hand), showing calculator steps will count as "work".

1. Use key points to graph two periods of each function, by hand, using key points. (8 points each)

a. $y = -3\sin\frac{\pi x}{2}$ Period $= \frac{2\pi}{\pi/2} = 4$

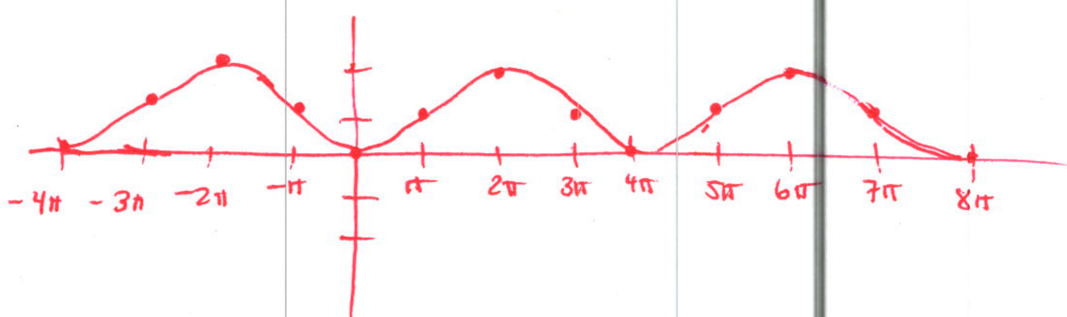


b. $y = \cos\left(\frac{1}{2}x - \pi\right) + 1$ Period $= \frac{2\pi}{1/2} = 4\pi$

amplitude $= 1$

phase shift is 2π

$\frac{1}{2}(x - 2\pi)$



2. For each function, state the amplitude, period, phase shift and any vertical shift. (4 points each)

a. $y = \frac{1}{2}\cos(2x + \pi)$

amplitude $= 1/2$

period $= \pi$

$2(x + \pi/2)$

phase shift left $\pi/2$

no vertical shift

b. $y = -5\sin\frac{2}{\pi}x + 2$

$\frac{2\pi}{2/\pi} = \pi^2$

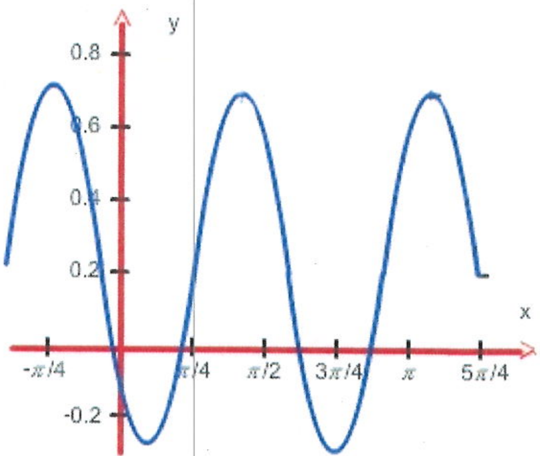
amplitude $= 5$

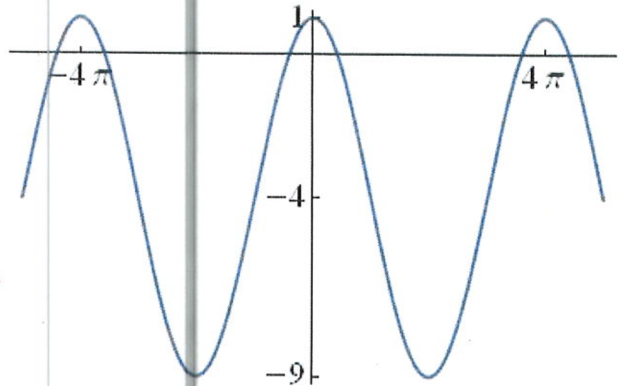
period $= \pi^2$

no phase shift

vertical shift up 2

3. Write an equation of the graph. (5 points each)





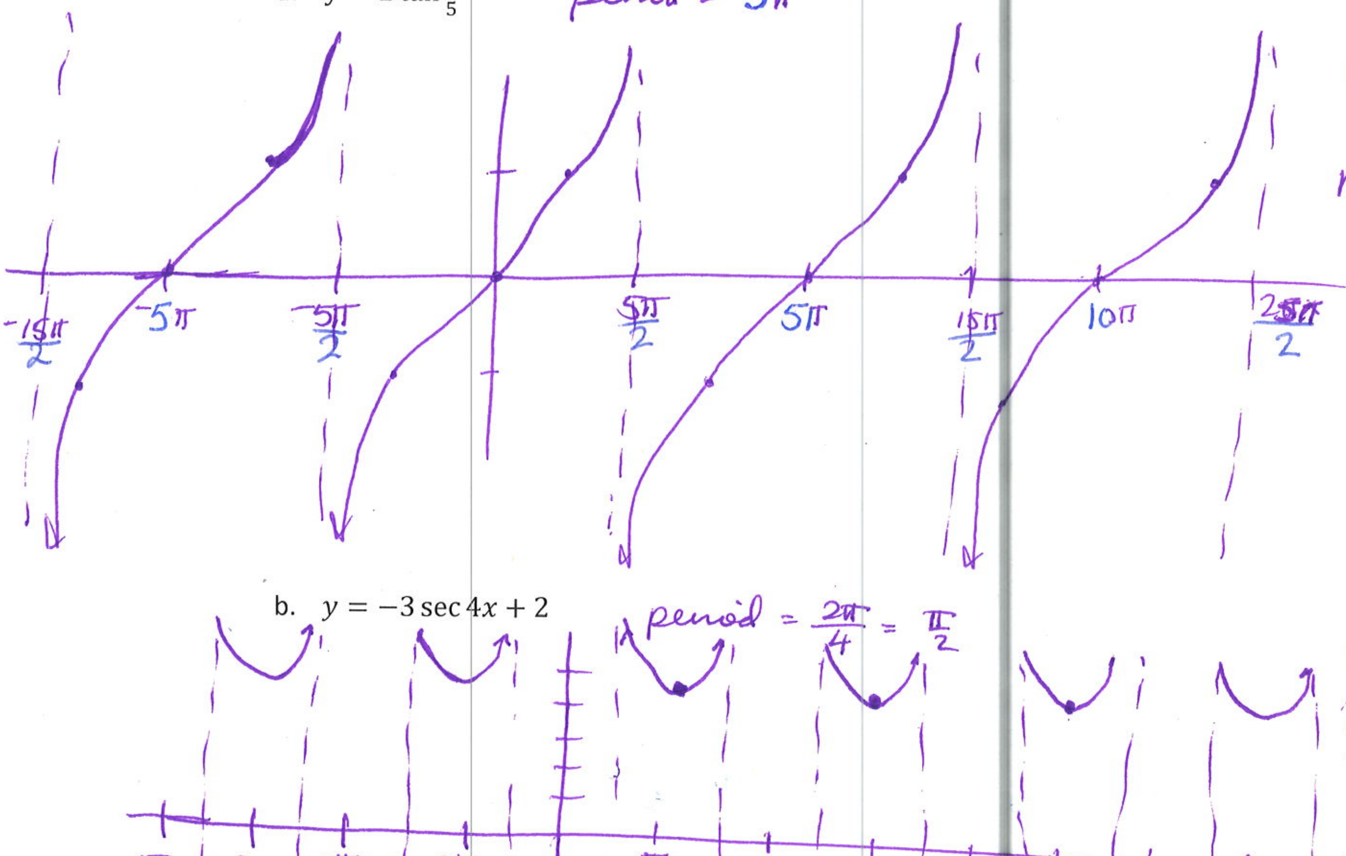
answers may vary

$y = \frac{1}{2}\cos\left[\frac{16}{5}\left(x + \frac{\pi}{4}\right)\right] + \frac{1}{5}$

$y = 5\cos\frac{1}{2}x - 4$

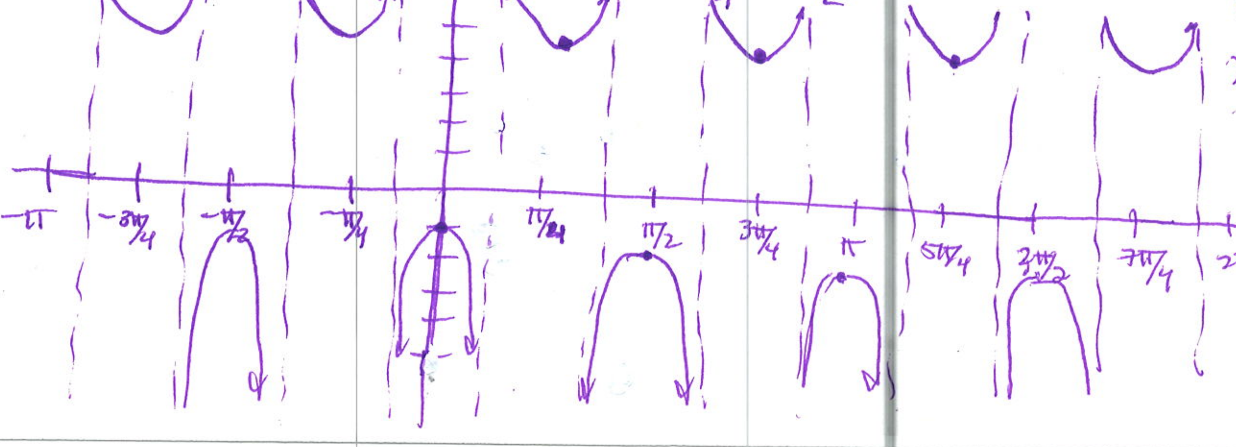
4. Graph the functions for 2 periods, by hand, using key points. State the domain of each. (7 points each)

a. $y = 2\tan\frac{x}{5}$ period $= 5\pi$



domain $x \neq$ odd multiples of $\frac{5\pi}{2}$

b. $y = -3\sec 4x + 2$ period $= \frac{2\pi}{4} = \frac{\pi}{2}$



domain: $x \neq$ odd multiples of $\pi/8$

5. Find the exact value of each expression. (4 points each)

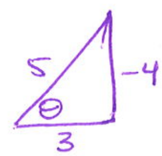
a. $\cos^{-1}\left(-\frac{1}{2}\right)$

$2\pi/3$

b. $\tan^{-1}\left(\tan\frac{5\pi}{4}\right)$

$\pi/4$

c. $\tan\left(sin^{-1}\left(-\frac{4}{5}\right)\right)$



$-\frac{4}{3}$

d. $\csc\left(cos^{-1}\left(-\frac{\sqrt{2}}{2}\right)\right)$

$\sqrt{2}$

6. Find the domain and range of $y = \sin^{-1}\left(\frac{x}{2}+1\right)+\frac{\pi}{3}$. (5 points)

[-1, 1] shift left by 1

horizontal stretch by 2 $\quad$ $[-2, 2] \rightarrow [-3, 1]$ domain

$[-\pi/2, \pi/2]$ shift up by $\pi/3$ $\rightarrow$ $[-\pi/6, 5\pi/6]$ range

7. Two triangles can be formed with $a = 30, b = 40, A = 15°$. Find all the missing sides and angles of both. (8 points)

$\frac{\sin 15°}{30} = \frac{\sin B}{40}$

$B = 20.2°$

$C = 144.8°$

$\frac{\sin 15°}{30} = \frac{\sin 144.8°}{c}$

$c = \frac{30 \sin 144.8}{\sin 15} = 66.8$

or

$B = 159.8°$

$C = 5.2°$

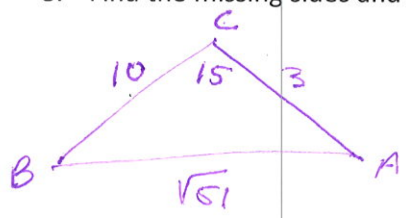
$\frac{\sin 15°}{30} = \frac{\sin 5.2°}{c}$

$c = \frac{30 \sin 5.2°}{\sin 15} = 10.5$

triangle 1: $a=30, b=40, c=66.8, A=15°, B=20.2°, C=144.8°$

triangle 2: $a=30, b=40, c=10.5, A=15°, B=159.8°, C=5.2°$

8. Find the missing sides and angles of a triangle with $a = 10, b = 3, C = 15°$. (7 points)



$a^2 + b^2 - 2ac\cos C = c^2$

$3^2 + 10^2 - 2(3)(10)\cos 15 = 51$

$C = \sqrt{51}$

$A = 21.2°$

$B = 143.8°$

$\frac{\sin 15}{\sqrt{51}} = \frac{\sin A}{10}$

$10\sin(15)/\sqrt{51} = \sin A$

9. Verify the identities. (6 points each)

a. $(\sec x - \tan x)^2 = \frac{1-\sin x}{1+\sin x}$

$\sec^2 x - 2\sec x\tan x + \tan^2 x = \frac{1}{\cos^2 x} - \frac{2}{\cos x}\frac{\sin x}{\cos x} + \frac{\sin^2 x}{\cos^2 x} =$

$\frac{1 - 2\sin x + \sin^2 x}{\cos^2 x} = \frac{(1-\sin x)^2}{1-\sin^2 x} = \frac{\cancel{(1-\sin x)}(1-\sin x)}{\cancel{(1-\sin x)}(1+\sin x)} = \frac{1-\sin x}{1+\sin x}$

b. $\frac{\sin^3 x - \cos^3 x}{\sin x - \cos x} = 1 + \sin x \cos x$

$\frac{\cancel{(\sin x - \cos x)}(\sin^2 x + \sin x\cos x + \cos^2 x)}{\cancel{\sin x - \cos x}} =$

$\sin^2 x + \cos^2 x + \sin x\cos x = 1 + \sin x\cos x$

c. $\sin\left(x + \frac{\pi}{2}\right) = \cos x$

$\sin x\,\cancel{\cos \pi/2}^{\,0} + \underbrace{\sin \pi/2}_{=1}\cos x = \cos x$

d. $\sin^2\frac{\theta}{2} = \frac{\sec\theta - 1}{2\sec\theta}$

$$= \frac{1-\cos\theta}{2}\left(\frac{\sec\theta}{\sec\theta}\right) = \frac{\sec\theta - 1}{2\sec\theta}$$

10. Use identities to find exact values for each of the following. (4 points each)

a. $\cos\frac{5\pi}{12}\cos\frac{\pi}{12} - \sin\frac{5\pi}{12}\sin\frac{\pi}{12}$

$$\cos\left(\frac{5\pi}{12} + \frac{\pi}{12}\right) = \cos\left(\frac{6\pi}{12}\right) = \cos\frac{\pi}{2} = \boxed{0}$$

b. $\sin 165°$

$$\sin(120° + 45°) = \sin 120° \cos 45° + \sin 45° \cos 120°$$

$$= \left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) + \left(\frac{\sqrt{2}}{2}\right)\left(-\frac{1}{2}\right) = \boxed{\frac{\sqrt{6} - \sqrt{2}}{4}}$$

c. $\cos\frac{\pi}{8}$ $= \sqrt{\frac{1 + \cos \pi/4}{2}} = \sqrt{\frac{1 + \frac{\sqrt{2}}{2}}{2}} = \sqrt{\frac{2+\sqrt{2}}{4}} = \boxed{\frac{\sqrt{2+\sqrt{2}}}{2}}$

d. $\tan\frac{7\pi}{12}$ $\left(\frac{7\pi}{12}\right)(2) = \frac{7\pi}{6}$

$$\frac{1 - \cos\frac{7\pi}{6}}{\sin 7\pi/6} = \frac{1 - \left(-\frac{\sqrt{3}}{2}\right)}{-1/2} = \frac{1 + \frac{\sqrt{3}}{2}}{-1/2} =$$

$$\left(1 + \frac{\sqrt{3}}{2}\right)\left(-\frac{2}{1}\right) = -2 - \sqrt{3}$$

11. Solve for all values of the variable in $[0,2\pi)$. (5 points each)

a. $\sin 2x = \cos x$

$\boxed{\left\{ \pi/6, \pi/2, 5\pi/6, 3\pi/2 \right\}}$

$2\sin x\cos x - \cos x = 0$

$\cos x(2\sin x - 1) = 0$

$\cos x = 0 \quad x = \pi/2, 3\pi/2$

$2\sin x - 1 = 0$

$\sin x = 1/2$

$x = \pi/6, \frac{5\pi}{6}$

b. $\sin\left(3x - \frac{\pi}{6}\right) = -\frac{\sqrt{2}}{2}$

$\sin\alpha = \frac{-\sqrt{2}}{2}$

$\alpha = \frac{5\pi}{4}, \frac{7\pi}{4}$, cycles 2π

$3x - \frac{\pi}{6} = \frac{5\pi}{4}$ cycles $\frac{2\pi}{3}$

$3x = \frac{17\pi}{12} \quad x = \frac{17\pi}{36}, \frac{41\pi}{36}, \frac{65\pi}{36}$

$3x - \frac{\pi}{6} = \frac{7\pi}{4}$

$3x = \frac{23\pi}{12} \Rightarrow x = \frac{23\pi}{36}, \frac{47\pi}{36}, \frac{71\pi}{36}$

$\boxed{\left\{ \frac{17\pi}{36}, \frac{23\pi}{36}, \frac{41\pi}{36}, \frac{47\pi}{36}, \frac{65\pi}{36}, \frac{71\pi}{36} \right\}}$

c. $3\cos^2 x = \sin^2 x$

$3\cos^2 x = 1 - \cos^2 x$

$4\cos^2 x = 1$

$\cos^2 x = \frac{1}{4}$

$\cos x = \pm\frac{1}{2}$

$x = \left\{ \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3} \right\}$

12. Find the equation of the ellipse in standard form if the endpoints of the major axis are (2,2) and (8,2), and one endpoint of the minor axis is (5,3). (7 points)

$\frac{2+8}{2} = (5,2)$ center $a=3$, $b=1$

$$\frac{(x-5)^2}{9} + \frac{(y-2)^2}{1} = 1$$

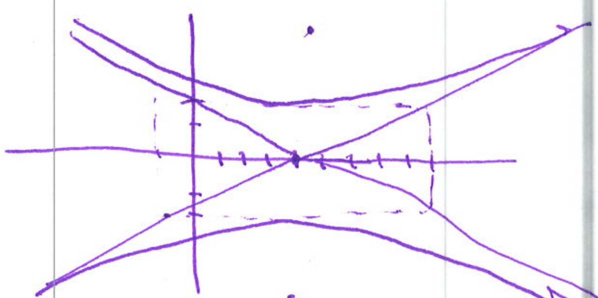
13. Find the equation of the conic $4x^2 - 25y^2 - 32x + 164 = 0$ in standard form, and sketch the graph. Label the center, the transverse axis, directrix, asymptotes, vertices, foci, etc. as it applies. (8 points)

$$4(x^2 - 8x + 16) - 25(y^2) = -164 + 64$$

$$\frac{4(x-4)^2}{-100} \frac{-25y^2}{-100} = \frac{-100}{-100} \Rightarrow \frac{y^2}{4} - \frac{(x-4)^2}{25} = 1$$

$4 + 25 = 29$

$c = \sqrt{29}$



center $(4, 0)$

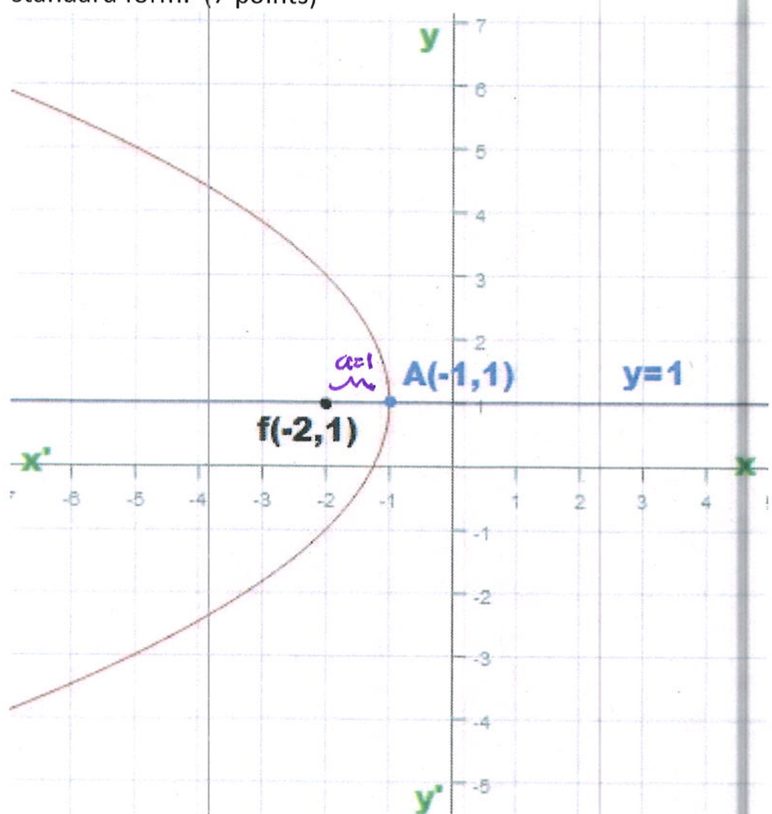
vertices $(4, 2), (4, -2)$

transverse axis $x = 4$

asymptotes $y = \frac{\pm 2}{5}(x-4)$

foci $(4, \sqrt{29}), (4, -\sqrt{29})$

14. Use the graph of the parabola shown with the labeled focus and vertex. Find the equation in standard form. (7 points)



$$(y-1)^2 = -4(x+1)$$

15. Simplify. $\dfrac{\frac{5}{x-3}-\frac{4}{x+3}}{\frac{10}{x^2-9}}$ $(x+3)(x-3)$ / $(x+3)(x-3)$ (5 points)

$$\frac{5(x+3)-4(x-3)}{10} = \frac{5x+15-4x+12}{10} = \boxed{\frac{x+27}{10}}$$

16. Solve $(3x-5)(x+2)=2$. Find all real or complex solutions. (6 points)

$3x^2+6x-5x-10=2$
$3x^2+x-12=0$

$$x=\frac{-1\pm\sqrt{1-4(3)(-12)}}{2(3)} = \boxed{\frac{-1\pm\sqrt{145}}{6}}$$

17. Solve.

a. $\left(\sqrt{x+10}\right)^2=(x-2)^2$ (6 points)

$x+10=x^2-4x+4$
$0=x^2-5x-6$
$(x-6)(x+1)=0$
$x=6,\ x=-1$

$\sqrt{6+10}=\sqrt{16}=4 \ ? \ 6-2=4$ ✓
$\sqrt{-1+10}=\sqrt{9}=3 \ ? \ -1-2=-3$ X

$\boxed{x=6}$

b. $\left(y-\frac{4}{y}\right)^2+5\left(y-\frac{4}{y}\right)-24=0$ (7 points)

$u^2+5u-24=0$
$(u+8)(u-3)=0$
$u=8,\ u=3$

$\left(y-\frac{4}{y}=8\right)y \Rightarrow y^2+8y-4=0$
$y=\frac{-8\pm\sqrt{64+16}}{2}=\frac{-8\pm4\sqrt{5}}{2}=-4\pm2\sqrt{5}$

$\left(y-\frac{4}{y}=3\right)y \Rightarrow y^2-3y-4=0$
$(y-4)(y+1)=0$
$y=4,\ y=-1$

$y=\{4,-1,-4\pm2\sqrt{5}\}$

18. Factor completely. (4 points each)

a. $2x^4 - 162$

$$2(x^4-81) = 2(x^2+9)(x^2-9) =$$
$$2(x^2+9)(x-3)(x+3)$$

b. $27x^3 - 125$

$$(3x-5)(9x^2+15x+25)$$

c. $(x+3)^{-5/2} - 4(x+3)^{-9/2}$

$$\frac{1}{(x+3)^{5/2}} - \frac{4}{(x+3)^{9/2}} = \frac{(x+3)^2-4}{(x+3)^{9/2}} = \frac{x^2+6x+9-4}{(x+3)^{9/2}} = \frac{\overbrace{x^2+6x+5}^{(x+5)(x+1)}}{(x+3)^{9/2}}$$

or

$$(x+1)(x+5)(x+3)^{-9/2}$$

19. Write a formula in summation notation for $\frac{1}{3}+\frac{2}{4}+\frac{3}{5}+\cdots+\frac{16}{18}$. (4 points)

$$\sum_{i=1}^{16} \frac{i}{i+2}$$

20. Write a formula for the infinite series $1+\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\cdots$, then find the sum. (6 points)

$$\sum_{i=0}^{\infty} \left(\frac{1}{3}\right)^i = \frac{1}{1-\frac{1}{3}} = \frac{1}{\frac{2}{3}} = \boxed{\frac{3}{2}}$$

21. For $f(x) = 4 - \frac{1}{x}$, and $g(x) = \frac{8}{x+7}$, find and simplify each of the following functions. State the domain of each. (5 points each)

a. $f - g$

$$4 - \frac{1}{x} - \frac{8}{x+7} = \frac{4x-1}{x} + \frac{-8}{x+7} = \frac{(4x-1)(x+7) - 8x}{x(x+7)}$$

$$\frac{4x^2 + 28x - x - 7 - 8x}{x(x+7)} = \boxed{\frac{4x^2 + 19x - 7}{x(x+7)}}$$

domain $x \neq 0, -7$

b. $\frac{f}{g}$

$$\frac{4 - \frac{1}{x}}{\frac{8}{x+7}} \cdot \frac{x(x+7)}{x(x+7)} = \frac{4x(x+7) - (x+7)}{8x} = \frac{4x^2 + 28x - x - 7}{8x}$$

$$= \boxed{\frac{4x^2 + 27x - 7}{8x}}$$

domain $x \neq 0,\ x \neq -7$

c. $g \circ f$

$$\frac{8}{\left(4 - \frac{1}{x}\right) + 7} = \frac{8}{11 - \frac{1}{x}}\left(\frac{x}{x}\right) = \frac{8x}{11x - 1}$$

domain: $x \neq 0, -\frac{1}{11}$

22. Find the inverse of $f(x) = \frac{x+2}{3x-1}$. State the domain of the inverse. (6 points)

$$x = \frac{y+2}{3y-1} \Rightarrow 3xy - x = y + 2 \Rightarrow 3xy - y = x + 2$$

$$y(3x-1) = x + 2$$

$$y = \frac{x+2}{3x-1} = f^{-1}(x)$$

domain of $f^{-1}(x)$

$x \neq 1/3$

yes, it is its own inverse

23. Find and simplify the difference quotient $\frac{f(x+h)-f(x)}{h}$ for $f(x) = x^2 - 3x + 7$. (7 points)

$$\frac{\cancel{x^2}+2xh+h^2-\cancel{3x}-3h+\cancel{7}-\cancel{x^2}+\cancel{3x}-\cancel{7}}{h} = \frac{2xh+h^2-3h}{h}$$

$$= \frac{\cancel{h}(2x+h-3)}{\cancel{h}} = \boxed{2x+h-3}$$

24. Use long division to divide the polynomials $\frac{x^5+2x^3-4x^2-5x-6}{x^2-x-2}$. Write the result at $q(x) + \frac{r(x)}{d(x)}$. (7 points)

$$\begin{array}{r} x^3+x^2+5x+3 \\ x^2-x-2 \overline{) x^5+0x^4+2x^3-4x^2-5x-6} \\ -x^5+x^4+2x^3 \\ \hline x^4+4x^3-4x^2-5x-6 \\ -x^4+x^3+2x^2 \\ \hline 5x^3-2x^2-5x-6 \\ -5x^3+5x^2+10x \\ \hline 3x^2+5x-6 \\ -3x^2+3x+6 \\ \hline 8x \end{array}$$

$$\boxed{x^3+x^2+5x+3+\frac{8x}{x^2-x-2}}$$

25. Use synthetic division to divide $\frac{x^6+x^5-10x^3+12}{x+2}$. Write the result at $q(x) + \frac{r(x)}{d(x)}$. (7 points)

$$\begin{array}{r|rrrrrrr} -2 & 1 & 1 & 0 & -10 & 0 & 0 & 12 \\ & & -2 & 2 & -4 & 28 & -56 & 112 \\ \hline & 1 & -1 & 2 & -14 & 28 & -56 & 124 \end{array}$$

$$x^5-x^4+2x^3-14x^2+28x-56+\frac{124}{x+2}$$

26. Use synthetic division and the Remainder Theorem to evaluate $f(x) = x^3 + 2x^2 - 5x - 3$ at $f(-2)$. (5 points)

$$\begin{array}{r|rrrr} -2 & 1 & 2 & -5 & -3 \\ & & -2 & 0 & 10 \\ \hline & 1 & 0 & -5 & 7 \end{array}$$

$f(-2) = 7$

27. Find all the rational zero of $f(x) = 2x^3 + x^2 - 3x + 1$ and use them to factor the polynomial completely to find all real and complex zeros. (7 points)

possible $\pm 1, \pm \frac{1}{2}$

$x = 1/2 \Rightarrow 2x - 1$

$$\begin{array}{r} x^2 + x - 1 \\ 2x-1 \overline{\big)\, 2x^3 + x^2 - 3x + 1} \\ -\,2x^3 + x^2 \\ \hline 2x^2 - 3x + 1 \\ -\,2x^2 + x \\ \hline -2x + 1 \\ -2x + 1 \\ \hline 0 \end{array}$$

$(2x-1)(x^2+x-1)$

$x = \dfrac{-1 \pm \sqrt{1+4}}{2} = \dfrac{-1 \pm \sqrt{5}}{2}$

$x = \left\{ \frac{1}{2}, \frac{-1 \pm \sqrt{5}}{2} \right\}$

28. Find a fourth degree polynomial with zeros $2, -4, 3 - 2i$, and $f(1) = -96$. (6 points)

$(x-2)(x+4)(x-3+2i)(x-3+2i)$

$\boxed{\frac{24}{11}(x^4 - 8x^3 - 7x^2 + 74x - 104) = f(x)}$

$(x-3+2i)(x-3-2i) = x^2 - 3x - \cancel{2ix} - 3x + 9 \cancel{+6i} + \cancel{2ix} \cancel{-6i} + 4$
$= x^2 - 6x + 13$

$(x^2 + 2x - 8)(x^2 - 6x + 13) = x^4 - 6x^3 + 13x^2 - 2x^3 - 12x^2 + 26x - 8x^2 + 48x - 104$

$a(x^4 - 8x^3 - 7x^2 + 74x - 104) = a(-44) = -96$

$x = 1$

$a = \frac{-24}{11}$

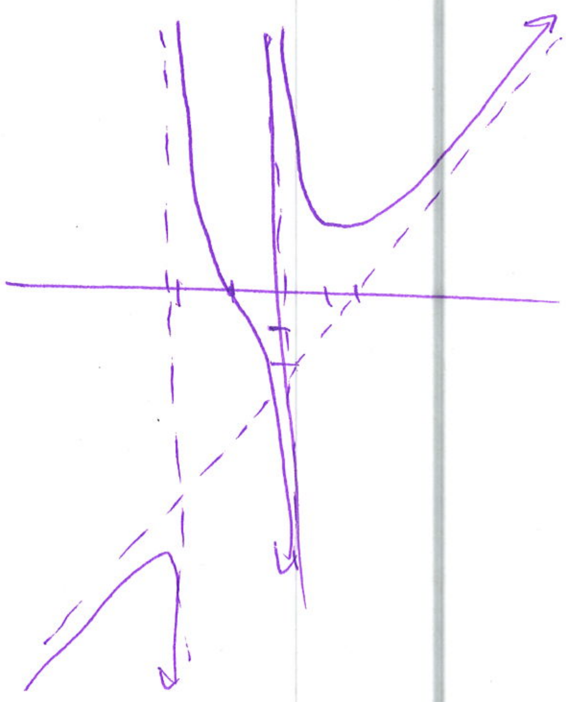
29. Find all vertical, horizontal or slant/oblique asymptotes, and any holes. Use that information to graph the function $f(x) = \frac{x^3+1}{x^2+2x}$. (8 points)

$$\begin{array}{r} x - 2 \\ x^2+2x \overline{\smash{\big)}\, x^3 + 0x^2 + 0x + 1} \\ -\ x^3 + 2x^2 \\ \hline -2x^2 + 0x + 1 \\ +2x^2 + 4x \\ \hline 4x + 1 \end{array}$$

$x - 2 + \frac{4x+1}{x^2+2x}$

vertical $x=0, x=-2$

oblique $y = x-2$



Some useful formulas:

$$\sin(a+b) = \sin a \cos b + \sin b \cos a$$
$$\sin(a-b) = \sin a \cos b - \sin b \cos a$$
$$\cos(a+b) = \cos a \cos b - \sin a \sin b$$
$$\cos(a-b) = \cos a \cos b + \sin a \sin b$$
$$\tan(a+b) = \frac{\tan a + \tan b}{1 - \tan a \tan b}$$
$$\tan(a-b) = \frac{\tan a - \tan b}{1 + \tan a \tan b}$$

$$\begin{aligned} \sin 2\alpha &= 2 \sin \alpha \cos \alpha \\ \cos 2\alpha &= \cos^2 \alpha - \sin^2 \alpha \\ &= 2\cos^2 \alpha - 1 \\ &= 1 - 2\sin^2 \alpha \\ \tan 2\alpha &= \frac{2 \tan \alpha}{1 - \tan^2 \alpha} \end{aligned}$$

$$\sin\left(\frac{a}{2}\right) = \pm\sqrt{\frac{(1-\cos a)}{2}}$$
$$\cos\left(\frac{a}{2}\right) = \pm\sqrt{\frac{(1+\cos a)}{2}}$$
$$\tan\left(\frac{a}{2}\right) = \frac{1-\cos a}{\sin a} = \frac{\sin a}{1+\cos a}$$

$$\sin^2 \theta = \frac{1-\cos 2\theta}{2}$$
$$\cos^2 \theta = \frac{1+\cos 2\theta}{2}$$
$$\tan^2 \theta = \frac{1-\cos 2\theta}{1+\cos 2\theta}$$