MAT 212, Quiz #15, Spring 2016 Name KEY

**Instructions**: Show all work. Answers without work required to obtain the solution will not receive full credit. Some questions may contain multiple parts: be sure to answer all of them. Give exact answers unless specifically asked to estimate.

1. Solve for the general solution for the first order homogeneous equation $\frac{dy}{dx} = \frac{x^2-3y^2}{2xy}$.

1st order homogeneous let $y = vx$ $y' = v'x + v$ $v = y/x$

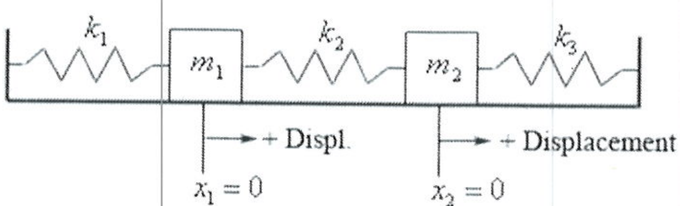
$$v'x + v = \frac{x^2+3v^2x^2}{2x^2v} = \frac{\cancel{x^2}(1+3v^2)}{\cancel{x^2}(2v)}$$

$$\frac{dv}{dx}\cdot x = \frac{1+3v^2 - v(2v)}{2v} = \frac{1+v^2}{2v} \Rightarrow \int \frac{2v}{1+v^2}\,dv = \int \frac{1}{x}\,dx$$

$$\ln(1+v^2) = \ln x + C \Rightarrow 1+v^2 = Ax$$

$$\Rightarrow \boxed{1 + \frac{y^2}{x^2} = Ax}$$

2. Set up the system of equations needed to solve the system of springs and masses displayed below if $k_1 = 3, k_2 = 4, k_3 = 2, m_1 = 1, m_2 = 2$. Do not solve.




$x_1' = x_3$ $x_3' = x_1''$
$x_2' = x_4$ $x_4' = x_2''$

$$m_1x_1'' = -k_1x_1 - k_2x_1 + k_2x_2$$
$$m_2x_2'' = -k_2x_2 - k_3x_2 + k_2x_1$$

$$\Rightarrow x_1'' = -(3+4)x_1 + 4x_2$$
$$\frac{2x_2''}{2} = \frac{4x_1 - (4+2)x_2}{2}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}' = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -7 & 4 & 0 & 0 \\ 2 & -3 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

3. Explain the conditions needed for resonance. Why is it a problem in a mechanical system?

undamped
forcing same as natural frequency

resonance is a problem because amplitude of oscillation increases over time leading to mechanical failure

4. Solve $xy'' + y = 0$ with a series solution centered at $x_0 = 1$.

$$y = \sum_{n=0}^{\infty} a_n (x-1)^n \quad y' = \sum_{n=1}^{\infty} a_n n (x-1)^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} a_n n(n-1)(x-1)^{n-2}$$

$$(x-1)y'' + y'' + y = 0$$

$$(x-1)\sum_{n=2}^{\infty} a_n n(n-1)(x-1)^{n-2} + \sum_{n=2}^{\infty} a_n n(n-1)(x-1)^{n-2} + \sum_{n=0}^{\infty} a_n (x-1)^n = 0$$

$$\sum_{n=2}^{\infty} a_n n(n-1)(x-1)^{n-1} + \sum_{n=2}^{\infty} a_n n(n-1)(x-1)^{n-2} + \sum_{n=0}^{\infty} a_n (x-1)^n = 0$$

$$\sum_{n=1}^{\infty} a_{n+1}(n+1)n(x-1)^n + \sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)(x-1)^n + \sum_{n=0}^{\infty} a_n (x-1)^n = 0$$

$$\sum_{n=1}^{\infty} \left[a_{n+1}(n+1)n + a_{n+2}(n+2)(n+1) + a_n\right](x-1)^n + a_2(2)(1) + a_0 = 0$$

$$a_{n+1}(n+1)n + a_{n+2}(n+2)(n+1) + a_n = 0 \qquad \frac{2a_2}{2} = -\frac{a_0}{2} \Rightarrow \boxed{a_2 = -\tfrac{1}{2}a_0}$$

5. Verify that $\vec{\Psi}(t) = \begin{pmatrix} e^{-3t} & e^{2t} \\ -4e^{-3t} & e^{2t} \end{pmatrix}$ is a solution to the system $\vec{\Psi}' = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix}\vec{\Psi}$.

$$\Psi' = \begin{pmatrix} -3e^{-3t} & 2e^{2t} \\ 12e^{-3t} & 2e^{2t} \end{pmatrix} \stackrel{?}{=} \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix}\begin{pmatrix} e^{-3t} & e^{2t} \\ -4e^{-3t} & e^{2t} \end{pmatrix} = \begin{pmatrix} e^{-3t} - 4e^{-3t} & e^{2t} + e^{2t} \\ 4e^{-3t} + 8e^{-3t} & 4e^{2t} - 2e^{2t} \end{pmatrix}$$

$$= \begin{pmatrix} -3e^{-3t} & 2e^{2t} \\ 12e^{-3t} & 2e^{2t} \end{pmatrix} \quad \checkmark \text{ yes, it is a sol.}$$

$$a_{n+2} = \frac{-a_n - a_{n+1}(n+1)n}{(n+2)(n+1)} = \frac{-a_n}{(n+2)(n+1)} - \frac{a_{n+1}n}{n+2}$$

$n=1$ $$a_3 = -\frac{a_1}{3\cdot 2} - \frac{a_2 \cdot 1}{3} = -\frac{1}{6}a_1 - \frac{1}{3}a_2 = -\frac{1}{6}a_1 - \frac{1}{3}\left(-\frac{1}{2}a_0\right) = -\frac{1}{6}a_1 + \frac{1}{6}a_0$$

$n=2$ $$a_4 = -\frac{a_2}{4\cdot 3} - \frac{a_3\cdot 2}{4} = -\frac{1}{12}\left(-\frac{1}{2}a_0\right) - \frac{1}{2}a_3 = \frac{1}{24}a_0 - \frac{1}{2}a_3 =$$
$$\frac{1}{24}a_0 - \frac{1}{2}\left(-\frac{1}{6}a_1 + \frac{1}{6}a_0\right) = -\frac{1}{24}a_0 + \frac{1}{12}a_1$$

$n=3$ $$a_5 = \frac{-a_3}{5\cdot 4} - \frac{a_4\cdot 3}{5} = -\frac{1}{20}\left(-\frac{1}{6}a_1 + \frac{1}{6}a_0\right) - \frac{1}{15}\left(-\frac{1}{24}a_0 + \frac{1}{12}a_1\right) = -\frac{1}{180}a_0 - \frac{1}{72}a_1$$

$$\boxed{\begin{aligned} y(x) = {} & a_0\left(1 - \tfrac{1}{2}(x-1)^2 + \tfrac{1}{6}(x-1)^3 - \tfrac{1}{24}(x-1)^4 - \tfrac{1}{180}(x-1)^5 + \cdots\right) + \\ & a_1\left((x-1) - \tfrac{1}{6}(x-1)^3 + \tfrac{1}{12}(x-1)^4 - \tfrac{1}{72}(x-1)^5 + \cdots\right) \end{aligned}}$$