MAT 212, Exam #2, Spring 2016 Name KEY

Instructions: Show all work. Answers without work required to obtain the solution will not receive full credit. Some questions may contain multiple parts: be sure to answer all of them. Give exact answers unless specifically asked to estimate.

1. Find the general solution for each of the following: (6 points each)

   a. $2y'' - 3y' + y = 0$

$2r^2 - 3r + 1 = 0$

$(2r - 1)(r - 1) = 0$

$r = 1/2 \quad r = 1$

$$\boxed{y(t) = c_1 e^{1/2 t} + c_2 e^{t}}$$

   b. $y'' + 2y' + 2y = 0$

$r^2 + 2r + 2 = 0$

$\frac{-2 \pm \sqrt{4-8}}{2} = \frac{-2 \pm 2i}{2} = -1 \pm i$

$$\boxed{y(t) = c_1 e^{-t} \cos t + c_2 e^{-t} \sin t}$$

   c. $y'' - 6y' + 9y = 0$

$r^2 - 6r + 9 = 0$

$(r-3)^2 = 0$

$r = 3$

$$\boxed{y(t) = c_1 e^{3t} + c_2 t e^{3t}}$$

d. $y^{VI} - y'' = 0$

$r^6 - r^2 = 0$

$r^2(r^4 - 1) = 0$

$r^2(r^2 - 1)(r^2 + 1)$

$r^2(r-1)(r+1)(r^2+1) = 0$

$r = 0$ repeated, $r = 1, -1, \pm i$

$$\boxed{y(t) = c_1 + c_2 t + c_3 e^{t} + c_4 e^{-t} + c_5 \cos t + c_6 \sin t}$$

e. $y^{IV} - 8y' = 0$

$r^4 - 8r = 0$

$r(r^3 - 8) = 0$

$r(r-2)(r^2 + 2r + 4) = 0$

$r = 0,\ r = 2 \quad r = -1 \pm \sqrt{3}i$

$$r = \frac{-2 \pm \sqrt{4 - 16}}{2} = \frac{-2 \pm \sqrt{-12}}{2}$$

$$= \frac{-2 \pm 2\sqrt{3}i}{2} = -1 \pm \sqrt{3}i$$

$$\boxed{y(t) = c_1 + c_2 e^{2t} + c_3 e^{-t}\cos\sqrt{3}t + c_4 e^{-t}\sin\sqrt{3}t}$$

2. Determine if the set of functions forms a fundamental set. (4 points each)

a. $e^t \sin t\,, e^t \cos t$

$$\begin{vmatrix} e^t \sin t & e^t \cos t \\ e^t \sin t + e^t \cos t & e^t \cos t - e^t \sin t \end{vmatrix} = \cancel{e^{2t} \sin t \cos t} - e^{2t} \sin^2 t - \cancel{e^{2t} \cos t \sin t} - e^{2t} \cos^2 t$$

$$-e^{2t}(\sin^2 t + \cos^2 t) = \boxed{-e^{2t}}$$

yes, a fundamental set

b. $e^t, \cosh t\,, \sinh t$

$$\begin{vmatrix} e^t & \cosh t & \sinh t \\ e^t & \sinh t & \cosh t \\ e^t & \cosh t & \sinh t \end{vmatrix} = e^t(\sin^2 t - \cosh^2 t) - \cosh t(e^t \sinh t - e^t \cosh t) +$$

$$\sinh t(e^t \cosh t - e^t \sinh t) =$$

$$e^t \cancel{\sinh^2 t} - e^t \cancel{\cosh^2 t} - e^t \cancel{\cosh t \sinh t} + e^t \cancel{\cosh^2 t} +$$

$$e^t \cancel{\sinh t \cosh t} - e^t \cancel{\sinh^2 t} = 0$$

not a fundamental set

c. $1, t, \cos t\,, \sin t$

$$\begin{vmatrix} 1 & t & \cos t & \sin t \\ 0 & 1 & -\sin t & \cos t \\ 0 & 0 & -\cos t & -\sin t \\ 0 & 0 & \sin t & -\cos t \end{vmatrix} = 1\begin{vmatrix} 1 & -\sin t & \cos t \\ 0 & -\cos t & -\sin t \\ 0 & \sin t & -\cos t \end{vmatrix} = (1)(1)\begin{vmatrix} -\cos t & -\sin t \\ \sin t & -\cos t \end{vmatrix} =$$

$$\cos^2 t + \sin^2 t = 1$$

fundamental set

3. Use Abel's Theorem to find the Wronskian for $\frac{x^2y''}{x^2} - \frac{x(x+2)y'}{x^2} + \frac{(x+2)y}{x^2} = 0$. (4 points)

$p(x) = \frac{x+2}{x} = 1 + \frac{2}{x}$

$W = e^{+\int 1+\frac{2}{x}\,dx} = e^{(x+2\ln x)} = e^{x}e^{\ln x^2} = x^2e^x * C_1$

4. Rewrite the expressions in standard form. (3 points each)

a. $e^{1+2i} = e^1e^{2i}$

$e\cos 2 + (e\sin 2)i$

c. $e^{(1-3i)x} = e^x \cdot e^{-3ix}$

$e^x\cos 3x - (e^x\sin 3x)i$

b. 2^{1-i} $e^{(\ln 2)(1-i)} = e^{\ln 2 - i\ln 2}$

$e^{\ln 2}e^{-i\ln 2} = 2e^{-i\ln 2}$

$2\cos(\ln 2) - 2\sin(\ln 2)i$

d. t^{4-i} $e^{(\ln t)(4-i)} = e^{4\ln t}e^{-i\ln t}$

$t^4 e^{-i\ln t}$

$t^4\cos(\ln t) - t^4\sin(\ln t)i$

5. Find the general solution for the Cauchy-Euler equation $t^2y'' + ty' + y = 0$. (7 points)

$y = t^n \quad y' = nt^{n-1} \quad y'' = n(n-1)t^{n-2}$

$t^2(n)(n-1)t^{n-2} + tnt^{n-1} + t^n = 0$

$n(n-1)t^n + nt^n + t^n = 0 \Rightarrow t^n[n^2 - \cancel{n} + \cancel{n} + 1] = 0 \quad n^2+1=0$

$n = \pm i \qquad t^i = \cos(\ln t) + \sin(\ln t)$

$\boxed{y(t) = c_1\cos(\ln t) + c_2\sin(\ln t)}$

6. Use reduction of order to find the second solution to the equation $(x-1)y'' - xy' + y = 0$, $y_1 = e^x$. (8 points)

$y_2 = ve^x \quad y_2' = v'e^x + ve^x \quad y_2'' = v''e^x + 2v'e^x + ve^x$

$(x-1)(v''e^x + 2v'e^x + ve^x) - x(v'e^x + ve^x) + ve^x = 0$

$xv''e^x + 2v'xe^x + \cancel{xve^x} - v''e^x - 2v'e^x - \cancel{ve^x} - v'xe^x - \cancel{xve^x} + \cancel{ve^x} = 0$

$xv'' + 2v'x - v'' - 2v' - v'x = 0$

$xv'' - v'' + v'x - 2v' = 0$

$v''(x-1) + v'(x-2) = 0 \qquad u = v' \quad u' = v''$

$\frac{u'(x-1)}{x-1} = -\frac{(x-2)}{x-1}u$

$$\begin{array}{r|l} & -1 \\ x-1 & -x+2 \\ & +x \mp 1 \\ \hline & 1 \end{array}$$

$\frac{du}{u} = \left(-1 + \frac{1}{x-1}\right)dx$

$\ln u = -x + \ln(x-1) + C$

$\cancel{\ln} u = \cancel{\ln}(Ae^{-x}(x-1)$

$v' = Ae^{-x}(x-1)$

$v = \int Ae^{-x}(x-1)\,dx \qquad u = x-1 \quad dv = e^{-x} \qquad du = dx \quad v = -e^{-x}$

$= -(x-1)e^{-x} + \int e^{-x}dx = -(x-1)e^{-x} - e^{-x}$

$(-x + \cancel{1} - \cancel{1})e^{-x} =$

$V = Axe^{-x}$

$y_2 = Axe^{-x} \cdot e^x = \boxed{Ax}$

7. What Ansatz would you need to solve for the given forcing function $F(t)$ and the specified solutions $y_1(t)$, $y_2(t)$ to the second order ODE. (2 points each)

	$y_1(t)$	$y_2(t)$	$F(t)$	Ansatz
a.	$\sin t$	$\cos t$	$\frac{7}{2}\sin 4t$	$A\sin 4t + B\cos 4t$
b.	e^{-2t}	e^{-4t}	$0.1e^{-2t}\sin 4t$	$Ae^{-2t}\sin 4t + Be^{-2t}\cos 4t$
c.	e^t	e^{-2t}	$\frac{3}{2}(e^t + e^{-t})$ $3\cosh t$	$Ate^t + Be^{-t}$
d.	$\sin \pi t$	$\cos \pi t$	$2t - 1$	$At + B$

8. Find the particular solution $y'' + 2y' + 5y = 3\sin 2t$, $y(0) = 1$, $y'(0) = 3$. (8 points)

$r^2 + 2r + 5 = 0$

$\frac{-2 \pm \sqrt{4-20}}{2} = \frac{-2 \pm 4i}{2} = -1 \pm 2i$

$Y_h(t) = c_1e^{-t}\cos 2t + c_2e^{-t}\sin 2t$

$y_p(t) = c_1e^{-t}\cos 2t + c_2e^{-t}\sin 2t + \frac{3}{17}\sin 2t + \frac{12}{17}\cos 2t$

$Y(t) = A\sin 2t + B\cos 2t$

$Y'(t) = 2A\cos 2t - 2B\sin 2t$

$Y''(t) = -4A\sin 2t - 4B\cos 2t$

$-4A\sin 2t - 4B\cos 2t + 4A\cos 2t - 4B\sin 2t + 5A\sin 2t + 5B\cos 2t = 3\sin 2t$

$-4A - 4B + 5A = \quad A - 4B = 3$

$-4B + 4A + 5B = \quad -4A + B = 0$

$A = \frac{3}{17},\ B = \frac{12}{17}$

9. Use variation of parameters $Y(t) = -y_1 \int \frac{y_2 g}{W} dt + y_2 \int \frac{y_1 g}{W} dt$ to solve $y'' - y' - 2y = 2e^{-t}$. Write the complete general solution. (8 points)

$$Y(t) = -e^{2t} \int \frac{e^{-t} \cdot 2e^{-t}}{-3e^{t}} dt + e^{-t} \int \frac{e^{2t} \cdot 2e^{-t}}{-3e^{t}} dt$$

$$= \frac{2}{3}e^{2t} \int e^{-3t} dt + -\frac{2}{3}e^{-t} \int dt =$$

$$\frac{2}{3}e^{2t}\left(-\frac{1}{3}e^{-3t}\right) + -\frac{2}{3}e^{-t} \cdot t =$$

part of $y_h(t)$ $\rightarrow$ $-\frac{2}{9}e^{-t} - \frac{2}{3}e^{-t} \cdot t$ ← new

$$y_p(t) = c_1 e^{2t} + c_1 e^{-t} - \frac{2}{3} t e^{-t}$$

$r^2 - r - 2 = 0$

$(r-2)(r+1) = 0$ $r = 2, r = -1$

$y_1 = e^{2t}$ $y_2 = e^{-t}$

$$W = \begin{vmatrix} e^{2t} & e^{-t} \\ 2e^{2t} & -e^{-t} \end{vmatrix} =$$

$-e^{t} - 2e^{t} = -3e^{t}$

10. A spring with a 4-kg mass has natural length 1 m and is maintained stretched to a length of 1.3 m by a force of 24.3 N. If the spring is compressed to a length of 0.8 m and then released with zero velocity, set up the second order linear IVP needed to solve the system. (You don't need to solve it, just set it up.) (5 points)

$m = 4$

$F = kx$

$24.3 = k(.3)$

$k = 81$

$y(0) = -.2$ $y'(0) = 0$

$4y'' + 81y = 0$ assuming no damping

11. Each of the functions below is a solution to a spring problem. Classify the portion of each function which is transient, and characterize the damping of the system. How many times does the system cross the equilibrium? If there is no transient solution, does the system experience resonance or beats? (3 points)

a. $y(t) = \underbrace{4\sin t - 3\cos t}_{\text{natural}} + \underbrace{\frac{4}{3} t \sin t}_{\text{forcing}}$

all steady-state

no transient solution

no damping ∴ undamped

cross $y=0$ ∞'ly many times

b. $y(t) = e^{-t} + 3e^{-2t}$

all transient, overdamped never crosses zero

c. $y(t) = \underbrace{10e^{-\left(\frac{t}{3}\right)} - 2te^{-\left(\frac{t}{3}\right)}}_{\text{transient}} + \underbrace{\frac{1}{4}t}_{\text{forcing/ steady state}}$

critically damped

never crosses zero

d. $y(t) = \underbrace{e^{-2t}\sin\frac{t}{4} + 6e^{-2t}\cos\frac{t}{4}}_{\text{transient}} + \underbrace{\cos 4t}_{\text{forcing}}$

Crosses zero ∞'lly many times

underdamped

none of the systems have resonance or beats

12. What are the conditions for a beats phenomenon to arise in a solution to a second order ODE? (4 points)

generally system is undamped

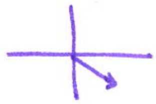
2 frequencies in system that have similar but not equal frequencies.

13. Find all the complex roots of the following: (4 points each)

a. $1^{\frac{1}{4}}$

$1, i, -1, -i$

c. $(1-i)^{\frac{1}{2}}$ $\sqrt{2}\,(e^{-\frac{\pi}{4}i})$



$\frac{7\pi}{4} \div 2 = \frac{7\pi}{8}$

$\frac{15}{4} \div 2 = \frac{15\pi}{8}$

$\sqrt[4]{2}\cos(\frac{7\pi}{8}) + i\sqrt[4]{2}\sin(\frac{7\pi}{8})$

$\sqrt[4]{2}\cos(\frac{15\pi}{8}) + i\sqrt[4]{2}\sin(\frac{15\pi}{8})$

b. $(1+\sqrt{3}i)^{\frac{1}{3}}$ $2e^{\frac{\pi}{3}i}$

$\frac{\pi}{9}, \frac{7\pi}{9}, \frac{13\pi}{9}$

$\sqrt[3]{2}\cos(\frac{\pi}{a}) + i\sqrt[3]{2}\sin(\frac{\pi}{a})$

$\sqrt[3]{2}\cos(\frac{7\pi}{a}) + i\sqrt[3]{2}\sin(\frac{7\pi}{a})$

$\sqrt[3]{2}\cos(\frac{13\pi}{a}) + i\sqrt[3]{2}\sin(\frac{13\pi}{a})$

14. Give an example of two functions that require variation of parameters to obtain the solution if they should appear in the forcing function of a second order or higher ODE, functions which cannot be solved by the method of undetermined coefficients. (2 points)

tan t,
Sec t
ln t

any function that does not have a finite set of derivatives