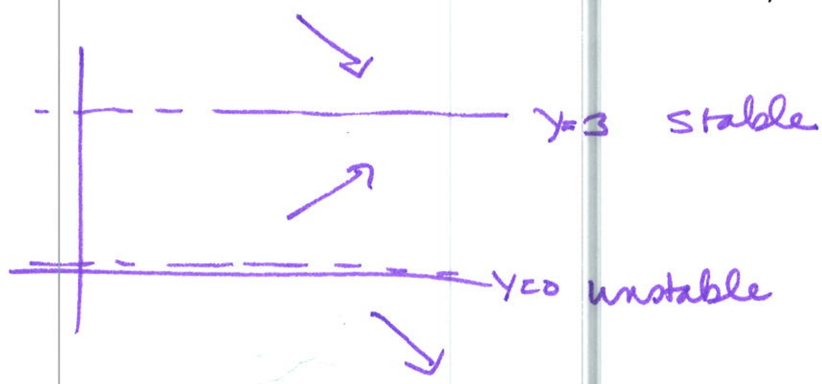
MAT 212, Exam #1, Spring 2016 Name KEY

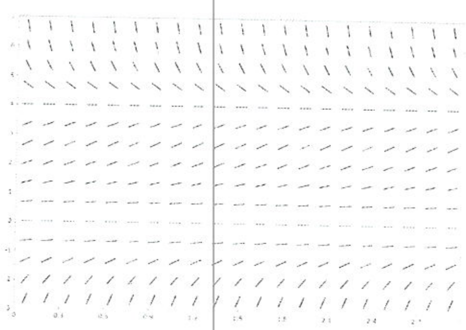
Instructions: Show all work. Answers without work required to obtain the solution will not receive full credit. Some questions may contain multiple parts: be sure to answer all of them. Give exact answers unless specifically asked to estimate.

1. Sketch the direction field for $y' = y(3-y)$. Label any equilibria and classify its stability.



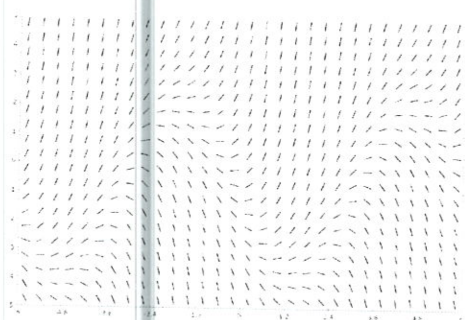
2. Match the direction fields to the corresponding differential equations.
   i. $y' = \frac{1}{4}y^2(4-y)$ a
   ii. $y' = 2t - 1 - y^2$ d
   iii. $y' = 3\sin(t) + 1 + y$ c
   iv. $y' = -\frac{2t+y}{2y}$ b

i



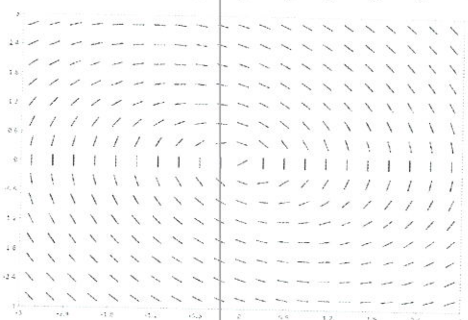
a.

iii



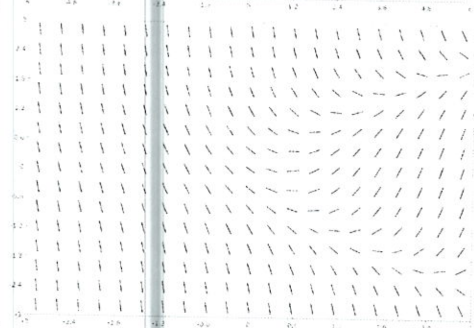
c.

iv.



b.

ii



d.

3. Classify the following differential equations by i) linearity, ii) order, iii) ordinary or partial.

a. $u_{xxxx} + 2u_{xxyy} + u_{yyyy} = 0$

linear, 4th order, partial

b. $u_t + uu_x = 1 + u_{xx}$

nonlinear, 2nd order, partial

c. $y''' - 3y'' + 2y' = 0$

linear, 3rd order, ordinary

d. $2t^2y'' + 3ty' - y = 0$

linear, 2nd order, ordinary

e. $\frac{d^3y}{dt^3} + t\frac{dy}{dt} + (\cos^2 y)t = t^3$

nonlinear, 3rd order, ordinary

4. Solve $2y' - y = e^{\frac{t}{3}}$ by the method of integrating factors.

$$y' - \frac{1}{2}y = \frac{1}{2}e^{t/3}$$

$$\mu = e^{\int -\frac{1}{2}dt} = e^{-\frac{1}{2}t}$$

$$e^{-\frac{1}{2}t}y' - \frac{1}{2}e^{-\frac{1}{2}t}y = \frac{1}{2}e^{-\frac{1}{6}t}$$

$$\int \left(e^{-\frac{1}{2}t}y\right)' = \int \frac{1}{2}e^{-\frac{1}{6}t}dt$$

$$e^{-\frac{1}{2}t}y = \frac{1}{2}(-6)e^{-\frac{1}{6}t} + C$$

$$\boxed{y = -3e^{t/3} + Ce^{\frac{1}{2}t}}$$

5. Solve $y' + y^2 \sin x = 0$ by separation of variables.

$$\frac{dy}{dx} = -y^2 \sin x$$

$$\int -\frac{dy}{y^2} = \int \sin x \, dx$$

$$\frac{1}{y} = -\cos x + C$$

$$\boxed{y(x) = \frac{1}{C - \cos x}}$$

6. Find the general solution for $y' = -\frac{4x+3y}{2x+y}$

$y = vx$
$y' = v'x + v$

$y' = v'x+v = -\frac{4x+3vx}{2x+vx} = -\frac{\cancel{x}(4+3v)}{\cancel{x}(2+v)}$

$v'x = \frac{-3v-4}{2+v} - v\frac{(2+v)}{(2+v)} = \frac{-3v-4-2v-v^2}{2+v} = \frac{-5v-4-v^2}{2+v}$

$\frac{2+v}{v^2+5v+4}dv = -\frac{1}{x}dx \Rightarrow \frac{2+v}{(v+1)(v+4)}dv = -\frac{1}{x}dx \Rightarrow \int \frac{-1}{v+1} + \frac{2}{v+4}dv = \int -\frac{1}{x}dx$

$\frac{A}{v+1} + \frac{B}{v+4} = \frac{Av+4A+Bv+B}{(v+1)(v+4)}$

$A+B=1$
$4A+B=2$

$A=-1$
$B=2$

$-\ln|v+1| + 2\ln|v+4| = -\ln x + C$

$\cancel{\ln}\left|\frac{(v+4)^2}{v+1}\right| = \cancel{\ln}\left(\frac{A}{x}\right) \Rightarrow \boxed{\frac{[(y/x)+4]^2}{(y/x)+1} = \frac{A}{x}}$

7. A tank has pure water flowing into it at 10 L/min. The contents of the tank are kept thoroughly mixed, and the contents flow out at 10 L/min. Salt is added to the tank at the rate of 0.1 kg/min. Initially, the tank contains 10 kg of salt in 100 L of water. How much salt is in the tank after 30 minutes?

$\frac{dQ}{dt}$ = Rate in − Rate out

Rate in $= \frac{10L}{min} \cdot \frac{0\ kg}{L} \Rightarrow 0 \quad +\frac{0.1\ kg}{min}$

Rate out $= \frac{10L}{min} \cdot \frac{Q}{100L} = \frac{Q}{10}$ $\quad Q(0)=10$

$\frac{dQ}{dt} = 0.1 - \frac{Q}{10} \Rightarrow Q' + \frac{1}{10}Q = .1$ $\quad \mu = e^{\int \frac{1}{10}dt} = e^{\frac{1}{10}t}$

$e^{+\frac{1}{10}t}Q' + \frac{1}{10}e^{+\frac{1}{10}t}Q = .1e^{+\frac{1}{10}t}$

$\int (e^{+\frac{1}{10}t}Q)' = \int .1e^{+\frac{1}{10}t}$

$e^{+\frac{1}{10}t}Q = e^{+\frac{1}{10}t} + C$

$Q = 1 + Ce^{-\frac{1}{10}t}$

$Q = 1 + 9e^{-\frac{1}{10}t}$

$C=9$

$Q(30) = 1+9e^{-\frac{1}{10}\cdot 30}$

$= \boxed{1.45\ kg}$

8. State the intervals over which the ODE $(4-t^2)y' + 2ty = 3t^2$ is guaranteed to have a solution.

$y' + \frac{2t}{4-t^2}y = \frac{3t^2}{4-t^2}$

$t \neq 2, -2$

defined on $(-\infty,-2) \cup (-2,2) \cup (2,\infty)$

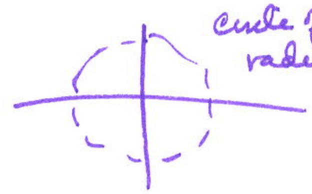
9. For the nonlinear ODE $y' = \frac{t-y}{1-t^2-y^2}$, sketch the region(s) in the plane where a solution is guaranteed to exist. Be sure to explicitly check all conditions required for these conditions to hold.

$$1-t^2-y^2=0$$
$$t^2+y^2=1$$

$$f = \frac{t-y}{1-t^2-y^2}$$

$$\frac{\partial f}{\partial y} = \frac{-1(1-t^2-y^2)-(t-y)(-2y)}{(1-t^2-y^2)^2} =$$

still not defined on same region



defined everywhere but on circle

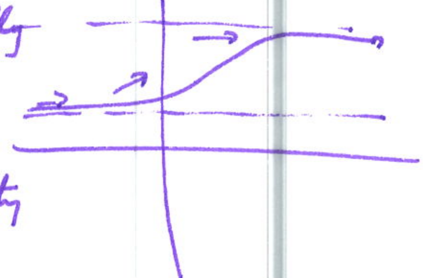
10. Rewrite the Bernoulli equation $y' + 2ty + ty^4 = 0$ as a first order linear equation. (You do not need to solve it.)

$$(1-n)y^{-n} = -3y^{-4}$$
$$z = y^{-3}$$
$$z' = -3y^{-4}y'$$

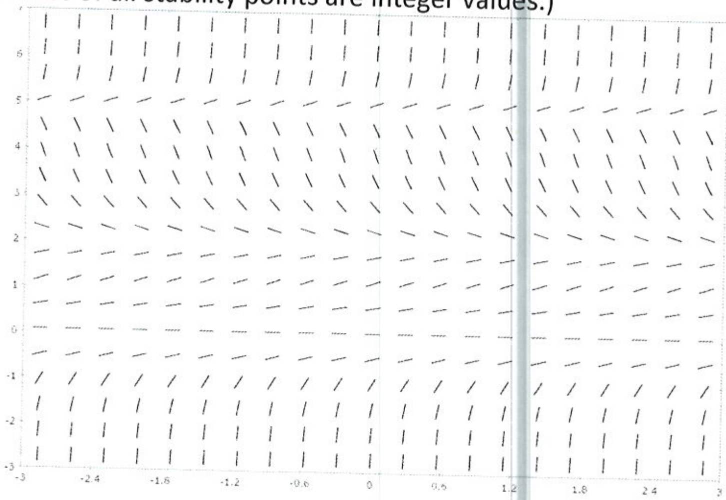
$$-3y^{-4}y' - 6ty^{-3} = 3t$$
$$\boxed{z' - 6tz = 3t}$$

11. Describe the behavior of a logistic solution to a population model. Sketch a graph of the direction field or the solution to illustrate.

logistic increases slowly at first then near exponentially then turns over and increases slowly as it approaches a carrying capacity (upper limit)



12. A direction field for a population model is shown. Write a differential equation to model the field. Characterize each equilibrium as a carrying capacity, threshold or neither. (You may assume that the values of all stability points are integer values.)



$\nearrow$

$y=5$ threshold

$\searrow$

$y=2$ carrying capacity

$\nearrow$

$y=0$ neither

$\nearrow$

$$\frac{dy}{dt} = y^2(y-2)(y-5)$$

13. Determine if the equation $\left(\frac{y}{x} + 6x\right) dx = (\ln x - 2)dy = 0$ is exact. If it is, solve it. If not, explain why not.

$$\frac{\partial M}{\partial y} = \frac{1}{x}$$

$$\frac{\partial N}{\partial x} = \frac{1}{x}$$

it is exact

$\int \frac{y}{x} + 6x \, dx = y\ln x + 3x^2 + f(y)$

$\int \ln x - 2 \, dy = y\ln x - 2y + g(x)$

$\Psi(x,y) = y\ln x + 3x^2 - 2y + k$

or $\boxed{y\ln x + 3x^2 - 2y = c}$

14. Use Euler's method to estimate $y(1)$ for the ODE $y' = \frac{4-ty}{1+y^2}, y(0) = -2$ in 5 steps.

$y_0 = -2 \quad t_0 = 0 \qquad m_1 = \frac{4-0(-2)}{1+4} = \frac{4}{5} \qquad y_1 = \frac{4}{5}(.2) + -2 \qquad \frac{1-0}{5} = .2$

$y_1 = -1.84 \quad t_1 = .2 \qquad m_2 = \frac{4-.2(-1.84)}{1+(1.84)^2} \qquad y_2 = .9959868661(.2) - 1.84$

$y_2 = -1.64\ldots \quad t_2 = .4 \qquad m_3 = \frac{4-.4(-1.64\ldots)}{1+(1.64)^2} \qquad y_3 = 1.26\ldots(.2) - 1.64$

$y_3 = -1.3885\ldots \quad t_3 = .6 \qquad m_4 = \frac{4-.6(-1.385\ldots)}{1+(-1.385)^2} \qquad y_4 = 1.65\ldots(.2) - 1.385$

$y_4 = -1.0584\ldots \quad t_4 = .8 \qquad m_5 = \frac{4-.8(-1.0584)}{1+(1.0584)^2} \qquad y_5 = 2.2858(.2) - 1.0584$

$y_5 = -.60129\ldots \quad t_5 = 1$

$\boxed{y(1) \approx -.6013}$