MAT 201, Final Exam, Spring 2016 Name KEY

Instructions: Show all work. Answers without work required to obtain the solution will not receive full credit. Some questions may contain multiple parts: be sure to answer all of them. Give exact answers unless specifically asked to estimate.

1. Find the average value of $f(x, y) = 5 - xy$ over $0 \le x \le 5, 0 \le y \le 1$. (8 points)

$A = 5 \times 1 = 5$

$$\frac{1}{5}\int_0^5\int_0^1 5 - xy\,dy\,dx = \frac{1}{5}\int_0^5 5y - \frac{1}{2}xy^2\Big|_0^1 = \frac{1}{5}\int_0^5 5 - \frac{1}{2}x\,dx =$$

$$\frac{1}{5}\left[5x - \frac{1}{4}x^2\right]_0^5 = \frac{1}{5}\left[25 - \frac{1}{4}\cdot 25\right] = 5 - \frac{5}{4} = \frac{20}{4} - \frac{5}{4} = \boxed{\frac{15}{4}}$$

2. Find the volume of the largest rectangular box in the 1st octant with faces of the coordinate planes and one vertex in the plane $x + 3y + 2z = 6$. (10 points)

$\boxed{(2, 2/3, 1)} \Rightarrow 4/3$

max. $f(x,y,z) = xyz$ s.t. $x + 3y + 2z = 6$ $\quad g(x,y,z) = x + 3y + 2z - 6 = 0$

$\nabla f = \langle yz, xz, xy \rangle \quad \lambda\nabla g = \lambda\langle 1, 3, 2\rangle = \langle \lambda, 3\lambda, 2\lambda \rangle$

$yz = \lambda$

$xz = 3\lambda = 3yz$
$xz - 3yz = 0$
$z(x - 3y) = 0$
$x = 3y \Rightarrow \frac{x}{3} = y$

$xy = 2\lambda = 2yz$
$xy - 2yz = 0$
$y(x - 2z) = 0$
$x = 2z \Rightarrow \frac{x}{2} = z$

$x + 3\left(\frac{x}{3}\right) + 2\left(\frac{x}{2}\right) = 6 \Rightarrow 3x = 6$
$x = 2$
$y = \frac{2}{3}$
$z = 1$

$V = (2)\left(\frac{2}{3}\right)(1) = 4/3$

3. Find the velocity and acceleration of a particle with a position function $\vec{r}(t) = (t^2 + t)\hat{\imath} + (t^2 - t)\hat{\jmath} + t^3\hat{k}$. (6 points)

$v(t) = (2t+1)\hat{\imath} + (2t-1)\hat{\jmath} + 3t^2\hat{k}$ $\quad (r'(t))$

$a(t) = 2\hat{\imath} + 2\hat{\jmath} + 6t\hat{k}$ $\quad (r''(t))$

4. Find the position vector of a particle with $\vec{a}(t) = 2t\hat{\imath} + \sin t\,\hat{\jmath} + \cos 2t\,\hat{k}$, $\vec{v}(0) = \hat{\imath}$, $\vec{r}(0) = \hat{\jmath}$. (10 points)

$v(t) = \int a(t)\,dt = (t^2 + c_1)\hat{\imath} + (-\cos t + c_2)\hat{\jmath} + (\tfrac{1}{2}\sin 2t + c_3)\hat{k}$

$=1 \quad c_1 = 1 \qquad =0 \quad -1 + c_2 = 0 \quad c_2 = 1 \qquad =0 \quad c_3 = 0$

$\boxed{v(t) = (t^2+1)\hat{\imath} + (-\cos t + 1)\hat{\jmath} + \tfrac{1}{2}\sin 2t\,\hat{k}}$

$r(t) = \int v(t)\,dt = (\tfrac{1}{3}t^3 + t + c_1)\hat{\imath} + (-\sin t + t + c_2)\hat{\jmath} + (-\tfrac{1}{4}\cos 2t + c_3)\hat{k}$

$=0 \quad c_1 = 0 \qquad =1 \quad c_2 = 1 \qquad =0 \quad -\tfrac{1}{4} + c_3 = 0 \quad c_3 = 1/4$

$\boxed{r(t) = (\tfrac{1}{3}t^3 + t)\hat{\imath} + (t + 1 - \sin t)\hat{\jmath} + (\tfrac{1}{4} - \tfrac{1}{4}\cos 2t)\hat{k}}$

5. Find the maximum or minimum value of the function $f(x,y) = x^2 + y^2 + 4x - 4y$ subject to the constraint $x^2 + y^2 \le 9$. (12 points)

$g = x^2 + y^2 - 9 = 0$

$\nabla f = \langle 2x+4,\ 2y-4 \rangle \qquad \lambda \nabla g = \langle 2x, 2y \rangle \cdot \lambda = \langle 2\lambda x, 2\lambda y \rangle$

$2x + 4 = 2\lambda x \qquad 2y - 4 = 2\lambda y$

$\frac{x+2}{x} = \frac{\lambda x}{x} \qquad \frac{y-2}{y} = \frac{\lambda y}{y}$

$\frac{x+2}{x} = \frac{y-2}{y} \Rightarrow \cancel{xy} - 2x = \cancel{xy} + 2y$

$-2x = 2y$

$x = -y$

$y = -x$

$x^2 + (-x)^2 \le 9$

$2x^2 \le 9$

$x^2 \le 9/2$

$x \le \pm 3/\sqrt{2}$

$y \ge \mp 3/\sqrt{2}$

max @ $(-\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2})$ and $(\frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{2})$

min @ $(0,0)$

6. Find the center of mass of the lamina bounded by $y = \sqrt{1-x^2}$ and $y = \sqrt{4-x^2}$ and the x-axis if the density is $\rho = kr$. (12 points)

$M = \int_0^{\pi}\int_1^2 kr \cdot r\,dr\,d\theta = \frac{k}{3}\int_0^{\pi} r^3\Big|_1^2\,d\theta =$

$\int_0^{\pi} \frac{k}{3}(8-1)\,d\theta = \frac{7k}{3}\cdot\pi = \frac{7k\pi}{3}$

$M_y = \int_0^{\pi}\int_1^2 kr\cdot r\cos\theta\cdot r\,dr\,d\theta = \int_0^{\pi}\frac{k}{4}r^4\Big|_1^2 \cos\theta\,d\theta = \frac{15k}{4}\int_0^{\pi}\cos\theta\,d\theta = \frac{15k}{4}\sin\theta\Big|_0^{\pi} = 0$

$16-1$

$M_x = \int_0^{\pi}\int_1^2 kr\cdot r\sin\theta\cdot r\,dr\,d\theta = \int_0^{\pi}\frac{k}{4}r^4\Big|_1^2\sin\theta\,d\theta = \frac{15k}{4}\int_0^{\pi}\sin\theta\,d\theta = -\frac{15k}{4}\cos\theta\Big|_0^{\pi}$

$-\frac{15k}{4}(-1-1) = +\frac{15k}{2}$

$\bar{x} = \frac{M_y}{M} = 0 \qquad \bar{y} = \frac{M_x}{M} = \frac{15k}{2}\cdot\frac{3}{7k\pi} = \frac{45}{14\pi}$

$\boxed{(0, \tfrac{45}{14\pi})}$

7. Set up but do not evaluate the integrals needed to find the center of mass for the region bounded by $x^2 + y^2 + z^2 \le 1, z \ge 0$, for $\rho = \sqrt{x^2 + y^2 + z^2}$ in spherical coordinates. (10 points)

$M = \int_0^{\pi/2}\int_0^{2\pi}\int_0^1 \rho\cdot\rho^2\sin\varphi\,d\rho\,d\theta\,d\varphi$

$M_{yz} = \int_0^{\pi/2}\int_0^{2\pi}\int_0^1 \rho\cdot\rho\sin\varphi\cos\theta\cdot\rho^2\sin\varphi\,d\rho\,d\theta\,d\varphi$

$M_{xz} = \int_0^{\pi/2}\int_0^{2\pi}\int_0^1 \rho\cdot\rho\sin\varphi\sin\theta\cdot\rho^2\sin\varphi\,d\rho\,d\theta\,d\varphi$

$M_{xy} = \int_0^{\pi/2}\int_0^{2\pi}\int_0^1 \rho\cdot\rho\cos\varphi\cdot\rho^2\sin\varphi\,d\rho\,d\theta\,d\varphi$

$\bar{x} = \frac{M_{yz}}{M} \qquad \bar{y} = \frac{M_{xz}}{M} \qquad \bar{z} = \frac{M_{xy}}{M}$

8. Find C for the joint density function $f(x, y, z) = Cxyz$ over the region $0 \le x \le 2, 0 \le y \le 2$, $0 \le z \le 2$ and zero otherwise. (12 points)

$$\int_0^2\int_0^2\int_0^2 Cxyz\,dz\,dy\,dx = \int_0^2\int_0^2 \frac{C}{2}xyz^2\Big|_0^2 dy\,dx = 2C\int_0^2\int_0^2 xy\,dy\,dx$$

$$= 2C\int_0^2 \frac{1}{2}xy^2\Big|_0^2 dx = 4C\int_0^2 x\,dx = 4C\left(\frac{1}{2}x^2\Big|_0^2 = 8C = 1 \Rightarrow C = \frac{1}{8}$$

$$f(x,y,z) = \frac{1}{8}xyz$$

9. Find the work done by the field $\vec{F}(x, y) = x^2\hat{\imath} + xy\hat{\jmath}$ as the particle moves around the circle $x^2 + y^2 = 4$ counterclockwise. (10 points)

$$\frac{\partial P}{\partial x} - \frac{\partial Q}{\partial y} = y - 0 = y$$

$$\int_0^{2\pi}\int_0^2 r^2\sin\theta\,dr\,d\theta = \int_0^{2\pi}\frac{1}{3}r^3\Big|_0^2 \sin\theta\,d\theta = \frac{8}{3}\int_0^{2\pi}\sin\theta\,d\theta =$$

$$\frac{8}{3}\left(-\cos\theta\Big|_0^{2\pi} = \frac{8}{3}(-1+1) = \boxed{0}$$

10. Find the mass of the wire with density $\rho = ze^{-xy}$ curved in the path $\vec{r}(t) = t\hat{\imath} + t^2\hat{\jmath} + e^{-t}\hat{k}$, $0 \le t \le 1$. (10 points)

$$M = \int_C \rho\,ds$$

$$\rho = e^{-t}\cdot e^{-\cancel{tt^2}} = \rho = e^{-t-t^3}$$

$$r'(t) = 1\hat{\imath} + 2t\hat{\jmath} + (-e^{-t})\hat{k} \quad ds = \|r'(t)\|dt = \sqrt{1+4t^2+e^{-2t}}\,dt$$

$$M = \int_0^1 e^{-t-t^3}\sqrt{1+4t^2+e^{-2t}}\,dt \approx .8208$$

11. Find the unit tangent vector for $\vec{r}(t) = 2\sqrt{t}\hat{\imath} + e^t\hat{\jmath} + e^{-t}\hat{k}$. (8 points)

$$r'(t) = \frac{1}{\sqrt{t}}\hat{\imath} + e^t\hat{\jmath} + (-e^{-t})\hat{k}$$

$$\|r'(t)\| = \sqrt{\frac{1}{t} + e^{2t} + e^{-2t}}$$

$$T(t) = \frac{\frac{1}{\sqrt{t}}\hat{\imath} + e^t\hat{\jmath} - e^{-t}\hat{k}}{\sqrt{\frac{1}{t} + e^{2t} + e^{-2t}}}$$

12. Find the function of the curvature for the equation $y = xe^x$. Then use it to estimate the radius of curvature at the point $x = 1$. (10 points)

$y' = e^x + xe^x = (1+x)e^x$

$y'' = e^x + e^x + xe^x = (2+x)e^x$

$R(1) \approx \frac{(1+4e^2)^{3/2}}{3e} \approx 20.71$

$K = \frac{|(2+x)e^x|}{(1+(1+x)^2e^{2x})^{3/2}}$ $\quad K(1) = \frac{3e}{(1+4e^2)^{3/2}}$

13. Sketch the gradient field of $f(x, y) = xy - 2x - 2y - x^2 - y^2$. Use the gradient field to sketch at least 5 level curves and locate any extrema. Based on the graph, determine if the extrema are maxima, minima, saddle points, or cannot be determined. Then use the second partials test on the same function to verify the results. (16 points)

$\nabla f = \langle y-2-2x,\ x-2-2y \rangle = \langle 0,0 \rangle$

$y = 2x+2 \quad x = 2y+2$

$y = 2(2y+2)+2$

$y = 4y+4+2$

$-3y = 6$

$y = -2 \quad x = -4+2 = -2$

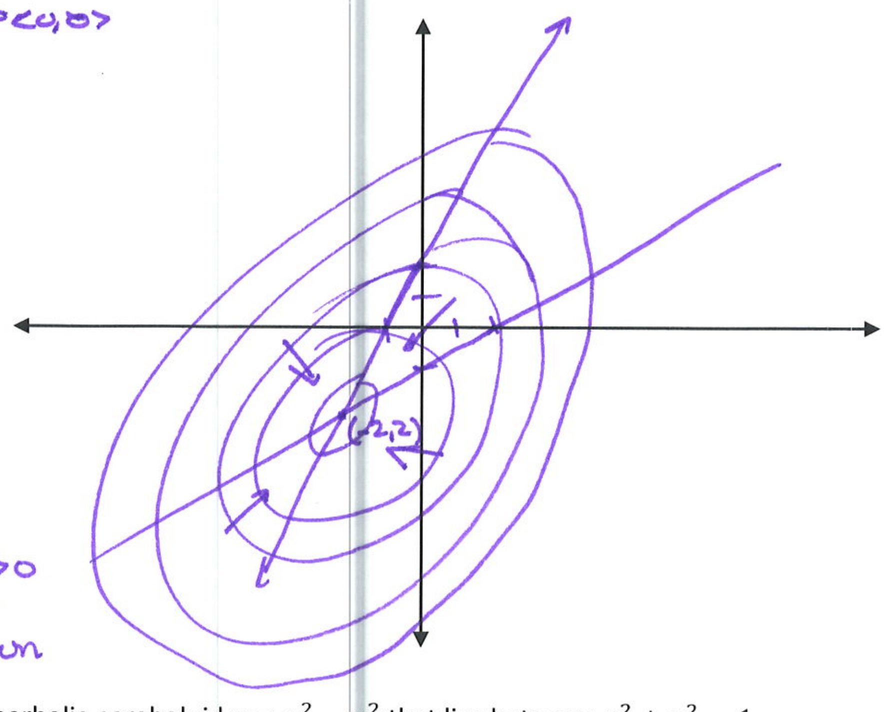
extrema $(-2,-2)$ max

$\nabla f(0,0) \quad \langle -2,-2 \rangle$

$\nabla f(-2,0) \quad \langle 2,-4 \rangle$

$\nabla f(-3,-3) \quad \langle 2,2 \rangle$

$\nabla f(0,-2) \quad \langle -4,2 \rangle$



$f_{xx} = -2$ $\quad D = (-2)(-2) - 1^2 = 3 > 0$

min or max

$f_{yy} = -2$

$f_{xy} = 1$ $\quad f_{xx} < 0$ concave down

max

14. Find the surface area of the hyperbolic paraboloid $z = x^2 - y^2$ that lies between $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$. (10 points)

$F(x,y,z) = x^2 - y^2 - z$

$\nabla F = \langle 2x, -2y, -1 \rangle \quad \|\nabla F\| = \sqrt{4x^2+4y^2+1} = \sqrt{4r^2+1}$

$\int_0^{2\pi}\int_1^2 r\sqrt{4r^2+1}\,dr\,d\theta =$

$u = 4r^2+1$

$du = 8r\,dr$

$\frac{1}{8}du = r\,dr$

$\int \frac{1}{8}u^{1/2}\,du$

$\frac{1}{8}\cdot\frac{2}{3}u^{3/2}$

$\int_0^{2\pi} \frac{1}{12}(4r^2+1)^{3/2}\Big|_1^2\,d\theta =$

$\int_0^{2\pi} \frac{1}{12}[17^{3/2} - 5^{3/2}]\,d\theta =$

$2\pi \cdot \frac{1}{12}[17^{3/2} - 5^{3/2}] = \boxed{\frac{\pi}{6}[17^{3/2} - 5^{3/2}]}$

15. Find the Jacobian of the transformation $x = e^{-r}\cos\theta$, $y = e^{r}\sin\theta$. (6 points)

$$\begin{vmatrix} -e^{-r}\cos\theta & -e^{-r}\sin\theta \\ e^{r}\sin\theta & e^{r}\cos\theta \end{vmatrix} = -\cos^2\theta + \sin^2\theta = \boxed{-\cos 2\theta}$$

16. Find an equation of the tangent plane to the surface $\vec{r}(u,v) = (1-u^2-v^2)\hat{\imath} - v\hat{\jmath} - u\hat{k}$ at the point $(-1,-1,-1)$. (10 points)

$u=1, v=1$

$\vec{r}_u = -2u\hat{\imath} + 0\hat{\jmath} - 1\hat{k}$ $\quad\vec{r}_v = -2v\hat{\imath} - 1\hat{\jmath} + 0\hat{k}$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -2u & 0 & -1 \\ -2v & -1 & 0 \end{vmatrix} = (0-1)\hat{\imath} - (0-2v)\hat{\jmath} + (2u-0)\hat{k}$$

$-\hat{\imath} + 2v\hat{\jmath} + 2u\hat{k} = \vec{n}$

$\langle -1, 2, 2 \rangle$

$$\boxed{-1(x+1) + 2(y+1) + 2(z+1) = 0}$$

17. Determine if $\vec{F}(x,y) = (ye^x + \sin y)\hat{\imath} + (e^x + x\cos y)\hat{\jmath}$ is conservative. (8 points)

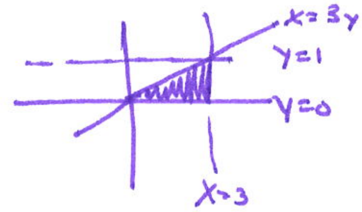
$\frac{\partial M}{\partial y} = e^x + \cos y$

$\frac{\partial N}{\partial x} = e^x + \cos y$

Same, so conservative

18. Evaluate $\int_0^1 \int_{3y}^3 e^{x^2}\, dxdy$ by sketching the region and changing the order of integration. (10 points)

$y=0,\ y=1$
$x=3,\ x=3y \Rightarrow y=\frac{x}{3}$



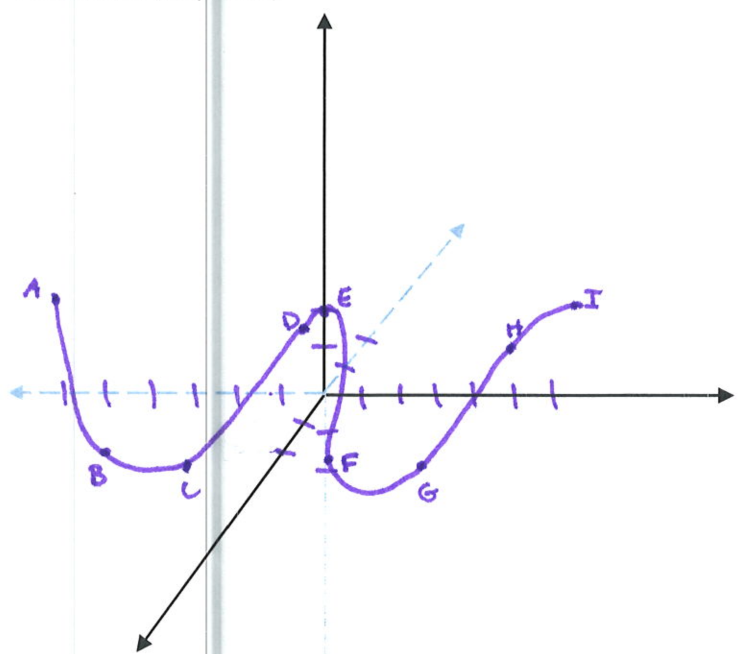
$$\int_0^3\int_0^{x/3} e^{x^2}\,dy\,dx = \int_0^3 \tfrac{1}{3}x e^{x^2}\,dx = \tfrac{1}{6}e^{x^2}\Big|_0^3$$

$\boxed{\frac{1}{6}[e^9-1]}$

$u=x^2$
$du=2x\,dx$
$\frac{1}{2}du = x\,dx$

19. Graph the space curve $\vec{r}(t) = 2\sin t\,\hat{\imath} + t\hat{\jmath} + 2\cos t\,\hat{k}$. (14 points)

t	x	y	z	
-2π	0	-6.28	2	A
$-3\pi/2$	2	-4.7	0	B
$-\pi$	0	-3.1	-2	C
$-\pi/2$	-2	-1.57	0	D
0	0	0	2	E
$\pi/2$	2	1.57	0	F
π	0	3.1	-2	G
$3\pi/2$	-2	4.7	0	H
2π	0	6.28	2	I



20. Find the limit. (7 points each)

a. $\lim\limits_{(x,y)\to(0,0)} \frac{2xy^2\cos^2 x}{x^3+y^3}$

$y=kx$

$$\lim_{x\to 0} \frac{2x\cdot k^2x^2\cos^2 x}{x^3+k^3x^3} =$$

$$\lim_{x\to 0} \frac{2k^2\cancel{x^3}\cos^2 x}{\cancel{x^3}(1+k^3)}$$

$$\lim_{x\to 0} \frac{2k^2\cos^2 x}{1+k^3} = \frac{2k^2}{1+k^3}$$

depends on k $\boxed{\text{DNE}}$

b. $\lim\limits_{(x,y)\to(0,0)} \frac{xy^2}{x^3+y^6}$

$y=k\sqrt{x}$

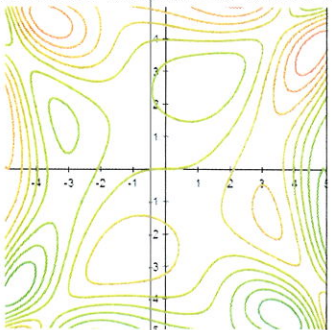
$$\lim_{x\to 0} \frac{xk^2x}{x^3+k^6x^3} =$$

$$\lim_{x\to 0} \frac{\cancel{x^2}\cdot k^2}{x^{\cancel{3}}(1+k^6)} = \lim_{x\to 0} \frac{k^2}{x(1+k^6)} = \pm\infty$$

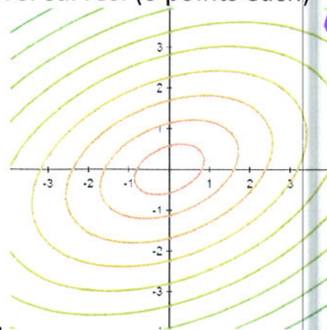
$\boxed{\text{DNE}}$

21. Match each function to its set of level curves. (6 points each)

V

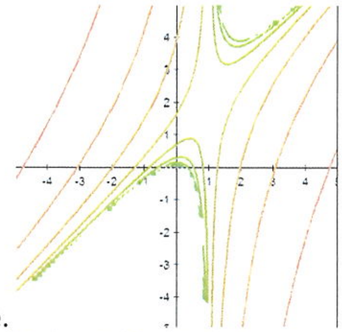


a.



c.

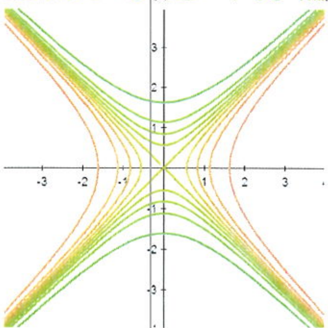
i



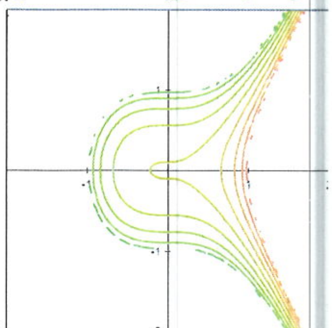
e.

iii

vi

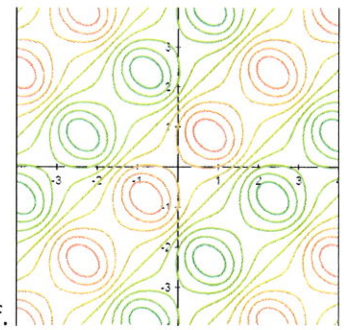


b.



d.

ii



f.

N

i. $z = 2 - \sqrt{x^2 - xy + 2y^2}$ c

ii. $z = \sin^{-1}(x^3 - y^2)$ d

iii. $z = \ln(x^2 - xy + y)$ e

iv. $z = e^{sin(x+y)^2} cos(x - y)$ f

v. $z = xsin\left(\frac{x^2 - y^2}{5}\right) - ycos\left(\frac{x^2 + y^2}{10}\right)$ a

vi. $z = \tan^{-1}(x^2 - y^2)$ b