MAT 201, Exam #2, Spring 2016 Name KEY

Instructions: Show all work. Answers without work required to obtain the solution will not receive full credit. Some questions may contain multiple parts: be sure to answer all of them. Give exact answers unless specifically asked to estimate.

1. Use Stokes' Theorem to evaluate $\int_C \vec{F} \cdot d\vec{r}$ for $\vec{F}(x,y,z) = xy\hat{\imath} + 2z\hat{\jmath} + 3y\hat{k}$ where C is the intersection of the plane $x + z = 5$ and $x^2 + y^2 = 9$ oriented counterclockwise from above. (5 points)

$\langle 1, 0, 1 \rangle = \vec{n}$

$\nabla \times F = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xy & 2z & 3y \end{vmatrix} = (3-2)\hat{\imath} - (0-0)\hat{\jmath} + (0-x)\hat{k}$

$\hat{\imath} + 0\hat{\jmath} - x\hat{k}$

$(\nabla \times F) \cdot \vec{n} =$

$1 + (-x) = 1 - x$

$\int_0^{2\pi}\int_0^3 (1 - r\cos\theta)\, r\,dr\,d\theta = \int_0^{2\pi}\int_0^3 r - r^2\cos\theta\,dr\,d\theta$

$= \int_0^{2\pi} \frac{1}{2}r^2 - \frac{1}{3}r^3\cos\theta \Big|_0^3 d\theta = \int_0^{2\pi} \frac{9}{2} - 9\cos\theta\,d\theta = \frac{9}{2}\theta - \cancel{9\sin\theta}^{\,0} \Big|_0^{2\pi} = \boxed{9\pi}$

2. Evaluate $\int_C \vec{F} \cdot d\vec{r}$ for $\vec{F}(x,y) = y\hat{\imath} + 2x\hat{\jmath}$ over the ellipse give by $x^2 + 2y^2 = 2$ [Recall: the area of an ellipse is $A = \pi ab$.] (5 points)

$\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} = 1 - 2 = -1$

$2 - 1 = 1$

$\iint_R -1\,dA = -1\,Area = \boxed{-\sqrt{2}\pi}$

or $\sqrt{2}\pi$

$\frac{x^2}{2} + \frac{y^2}{1} = 1$

$a^2 = 2 \Rightarrow a = \sqrt{2}$

$b^2 = 1 \Rightarrow b = 1$

$A = \sqrt{2}\pi$

3. Use the fact that $\vec{F}(x,y) = xy^2\hat{\imath} + x^2y\hat{\jmath}$ is conservative to evaluate $\int_C \vec{F} \cdot d\vec{r}$ on $C: \vec{r}(t) = \left[t + \sin\left(\frac{\pi}{2}t\right)\right]\hat{\imath} + \left[t + \cos\left(\frac{\pi}{2}t\right)\right]\hat{\jmath}, 0 \le t \le 1$. (5 points)

$\int xy^2\,dx = \frac{1}{2}x^2y^2 + \text{stuff}$ $\quad f = \frac{1}{2}x^2y^2$

$\int x^2y\,dy = \frac{1}{2}x^2y^2 + \text{stuff}$

$\vec{r}(0) = (0+0)\hat{\imath} + (0+1)\hat{\jmath} = (0,1)$

$\vec{r}(1) = (1+1)\hat{\imath} + (1+0)\hat{\jmath} = (2,1)$

$f(2,1) - f(0,1) = \frac{1}{2}(2)^2(1)^2 - \frac{1}{2}(0)^2(1)^2 = 2 - 0 = \boxed{2}$

4. Determine if $\vec{F}(x,y) = (ye^x + \sin y)\hat{\imath} + (e^x + x\cos y)\hat{\jmath}$ is conservative. (4 points)

$$\frac{\partial M}{\partial y} = e^x + \cos y$$

$$\frac{\partial N}{\partial x} = e^x + \cos y$$

yes, it is conservative since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

5. Evaluate $\int_C \vec{F} \cdot d\vec{r}$ for $\vec{F}(x,y,z) = x\hat{\imath} - y\hat{\jmath} + xy\hat{k}$ over the curve $C: \vec{r}(t) = \cos t\,\hat{\imath} + \sin t\,\hat{\jmath} + t\hat{k}, 0 \le t \le \pi$. (5 points)

$r'(t) = -\sin t\,\hat{\imath} + \cos t\,\hat{\jmath} + 1\,\hat{k} \cdot F(t) = \cos t\,\hat{\imath} - \sin t\,\hat{\jmath} + \cos t \sin t\,\hat{k}$

$-\sin t\cos t - \sin t\cancel{\cos t} + \cos t\cancel{\sin t} = -\sin t\cos t$

$$-\int_0^\pi \sin t\cos t\,dt = -\frac{1}{2}\sin^2 t\Big|_0^\pi = \boxed{0}$$

6. Evaluate $\int_C y\,dx + z\,dy + x\,dz$ on $C: \vec{r}(t) = \sqrt{t}\hat{\imath} + t\hat{\jmath} + t^2\hat{k}, 1 \le t \le 4$. (5 points)

$dx = \frac{1}{2\sqrt{t}}$ $dy = dt$ $dz = 2t\,dt$

$$\int_1^4 \frac{t}{2\sqrt{t}} + t^2 + 2t\sqrt{t}\,dt = \int_1^4 \frac{1}{2}t^{1/2} + t^2 + 2t^{3/2}\,dt =$$

$$\frac{1}{2}\cdot\frac{2}{3}t^{3/2} + \frac{1}{3}t^3 + 2\cdot\frac{2}{5}t^{5/2}\Big|_1^4 = \frac{1}{3}t^{3/2} + \frac{1}{3}t^3 + \frac{4}{5}t^{5/2}\Big|_1^4 =$$

$$\frac{1}{3}(8) + \frac{1}{3}(64) + \frac{4}{5}(32) - \frac{1}{3} - \frac{1}{3} - \frac{4}{5} = \boxed{\frac{722}{15}}$$

7. Evaluate $\int_C x\sin y\,ds$ on the line segment from $(0,3)$ to $(4,6)$. (5 points)

$\langle 4, 3\rangle$ $0 \le t \le 1$

$r(t) = 4t\,\hat{\imath} + (3t+3)\hat{\jmath}$

$r'(t) = \langle 4,3\rangle$ $\|r'(t)\| = \sqrt{16+9} = 5$

$$\int_0^1 4t\cdot\sin(3t+3)\cdot 5\,dt =$$

$$20\int_0^1 t\sin(3t+3)\,dt$$

$u = t$ $dv = \sin(3t+3)$

$du = dt$ $v = -\frac{1}{3}\cos(3t+3)$

$$20\left[-\frac{t}{3}\cos(3t+3) + \int\frac{1}{3}\cos(3t+3)\,dt\right] = 20\left[-\frac{t}{3}\cos(3t+3) + \frac{1}{9}\sin(3t+3)\Big|_0^1\right] =$$

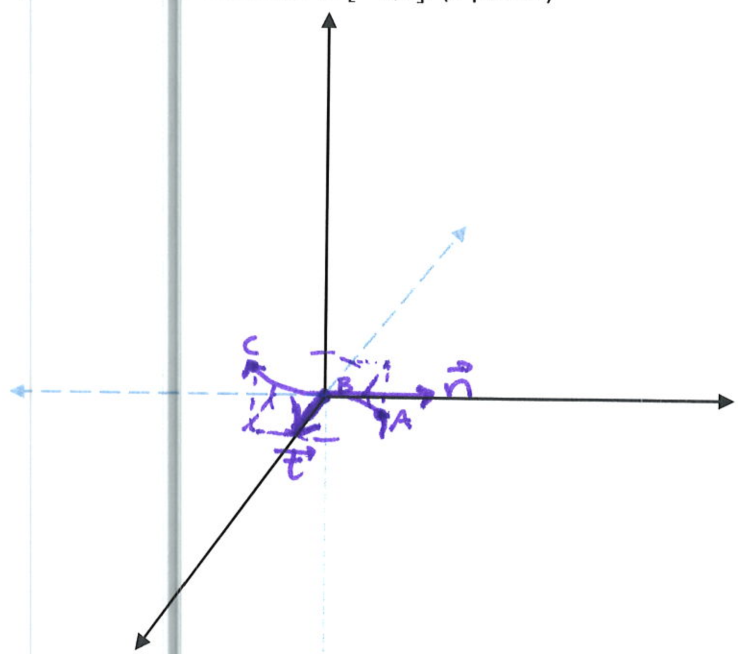
$$20\left[-\frac{1}{3}\cos(6) + \frac{1}{9}\sin 6 + 0 - \frac{1}{9}\sin(3)\right] = \boxed{\frac{20}{9}\sin 6 - \frac{20}{3}\cos 6 - \frac{20}{9}\sin 3}$$

8. Sketch the graph of the equation $\vec{r}(t) = t\hat{\imath} + t^2\hat{\jmath} + t^3\hat{k}$ on the interval $t \in [-1,1]$. (3 points)

   a. At the point $t = 0$, sketch the tangent vector and the normal vector. (2 points)

$t=-1 \quad \langle -1, 1, -1 \rangle$

$t=1 \quad \langle 1, 1, 1 \rangle$



   b. Find each of the following: (2 points)

      i. $\vec{r}'(t)$

$\langle 1, 2t, 3t^2 \rangle$

      ii. $\vec{r}''(t)$

$\langle 0, 2, 6t \rangle$

      iii. $\vec{r}'(t) \times \vec{r}''(t)$

$$\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 2t & 3t^2 \\ 0 & 2 & 6t \end{vmatrix} = (12t^2-6t^2)\hat{\imath} - (6t-0)\hat{\jmath} + (2-0)\hat{k}$$

$$6t^2\hat{\imath} - 6t\hat{\jmath} + 2\hat{k}$$

      iv. $\vec{r}'(t) \cdot \vec{r}''(t)$

$0 + 4t + 18t^3 = 18t^3 + 4t$

   c. Find the equation of the tangent line at the point $t = 0$. (3 points)

$\langle 1, 0, 0 \rangle$

$\vec{r}(t) = t\hat{\imath}$

9. Integrate $\int_0^1 \left(\frac{4}{1+t^2}\hat{\jmath} + \frac{2t}{1+t^2}\hat{k}\right) dt$. (4 points)

$$(4\arctan t)\hat{\jmath} + \ln(1+t^2)\hat{k}\Big|_0^1$$

$$\left(4\tan^{-1}(1) - 4\underbrace{\tan^{-1}0}_{0}\right)\hat{\jmath} + \left(\ln(1+1) - \underbrace{\ln(1)}_{0}\right)\hat{k} =$$

$$\boxed{\pi\hat{\jmath} + \ln 2\,\hat{k}}$$

10. For the function $u(x,y,z) = xy\sin^{-1}(yz)$, find all three first partial derivatives. (6 points)

$$\frac{\partial u}{\partial x} = y\sin^{-1}(yz)$$

$$\frac{\partial u}{\partial y} = x\sin^{-1}(yz) + xy \cdot \frac{1}{\sqrt{1-y^2z^2}} \cdot z = x\sin^{-1}(yz) + \frac{xyz}{\sqrt{1-y^2z^2}}$$

$$\frac{\partial u}{\partial z} = xy \cdot \frac{1}{\sqrt{1-y^2z^2}} \cdot y = \frac{xy^2}{\sqrt{1-y^2z^2}}$$

11. For $u = e^{xy}\sin y$, verify that $u_{xy} = u_{yx}$. (5 points)

$u_x = ye^{xy}\sin y$ $\quad u_{xy} = e^{xy}\sin y + xye^{xy}\sin y + ye^{xy}\cos y$

$u_y = xe^{xy}\sin y + e^{xy}\cos y$ $\quad u_{yx} = e^{xy}\sin y + xye^{xy}\sin y + ye^{xy}\cos y$

These are equal

12. Evaluate the following for $f(x,y) = y^2e^{xyz}$ and $\vec{F}(x,y,z) = 3xy^2\hat{\imath} + 5yz\hat{\jmath} + 2xz^3\hat{k}$. (3 points each)

a. $\vec{\nabla} f$

$$\langle y^2 \cdot yze^{xyz},\ 2ye^{xyz} + y^2e^{xyz} \cdot xz,\ y^2 \cdot xye^{xyz} \rangle$$

$$= \boxed{\langle y^3ze^{xyz},\ 2ye^{xyz} + xy^2ze^{xy^2},\ xy^3e^{xyz} \rangle}$$

b. $\vec{\nabla} \times \vec{F}$

$$\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 3xy^2 & 5yz & 2xz^3 \end{vmatrix} = (0-5y)\hat{\imath} - (2z^3-0)\hat{\jmath} + (0-6xy)\hat{k}$$

$$\boxed{-5y\hat{\imath} - 2z^3\hat{\jmath} - 6xy\hat{k}}$$

c. $\vec{\nabla} \cdot \vec{F}$

$$\boxed{3y^2 + 5z + 6xz^2}$$

d. $\nabla^2 f$

$$y^3z \cdot yze^{xyz} + 2e^{xyz} + 2y \cdot xze^{xyz} + 2xyze^{xyz} + xy^2zxze^{xyz} + xy^3xye^{xyz}$$

$$= e^{xyz}\left[y^4z^2 + 2 + 2xyz + 2xyz + x^2y^2z^2 + x^2y^4\right]$$

$$= \boxed{e^{xyz}\left[y^4z^2 + 2 + 4xyz + x^2y^2z^2 + x^2y^4\right]}$$

13. Evaluate the double integral $\iint_R 3dA$ over $R: -2 \le x \le 2, 1 \le y \le 6$ **geometrically**. What kind of shape is this? (4 points)

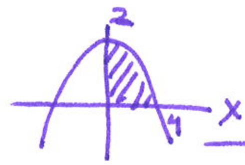
rectangular box

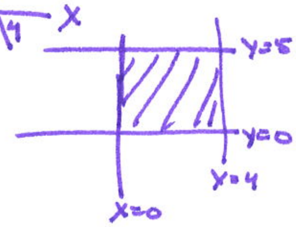
$4 \times 5 = 20$

$4 \times 5 \times 3 = \boxed{60}$

14. Find the volume of the indicated solid. (4 points each)

a. Volume in the 1[st] octant bounded by $z = 16 - x^2$ and $y = 5$.





$$\int_0^4 \int_0^5 \int_0^{16-x^2} dz\, dy\, dx =$$

$$\int_0^4 \int_0^5 16-x^2\, dy\, dx = 5\int_0^4 16-x^2\, dx =$$

$$5\left[16x - \tfrac{1}{3}x^3 \Big|_0^4\right] = 5\left[64 - \tfrac{64}{3}\right] = \boxed{\tfrac{640}{3}}$$

b. Volume bounded by coordinate planes and $3x + 2y + z = 6$.

$$\int_0^2 \int_0^{3-\frac{3}{2}x} \int_0^{6-3x-2y} dz\, dy\, dx =$$

$z = 6 - 3x - 2y$
$y = 3 - \frac{3}{2}x$

$$\int_0^2 \int_0^{3-\frac{3}{2}x} 6-3x-2y\, dy\, dx = \int_0^2 6y - 3xy - y^2 \Big|_0^{3-\frac{3}{2}x} dx = 6(3-\tfrac{3}{2}x) - 3x(3-\tfrac{3}{2}x) - (3-\tfrac{3}{2}x)^2\, dx$$

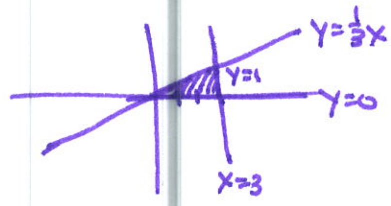
$-(9 - 9x + \frac{9}{4}x^2)$

$$\int_0^2 18 - 9x - \cancel{9x} + \tfrac{9}{2}x^2 - 9 + \cancel{9x} - \tfrac{9}{4}x^2 = \int_0^2 9 - 9x + \tfrac{9}{4}x^2\, dx =$$

$$9x - \tfrac{9}{2}x^2 + \tfrac{3}{4}x^3 \Big|_0^2 = 18 - 18 + 6 = \boxed{6}$$

15. Evaluate $\int_0^1 \int_{3y}^3 e^{x^2}\, dx\, dy$ by sketching the region and changing the order of integration. (5 points)

$y=1$
$y=0$
$x=3$
$x=3y$
$y=\frac{1}{3}x$

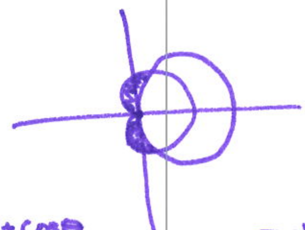


$$\int_0^3 \int_0^{1/3x} e^{x^2} dy\, dx = \int_0^3 ye^{x^2}\Big|_0^{1/3x} dx = \int_0^3 \tfrac{1}{3}xe^{x^2} dx$$

$u = x^2$
$du = 2x\,dx$
$\frac{1}{2}du = x\,dx$
$\frac{1}{3}\int \frac{1}{2}e^u\, du$
$= \frac{1}{6}e^u$

$$\tfrac{1}{6}e^{x^2}\Big|_0^3 = \tfrac{1}{6}(e^9 - e^0) = \boxed{\tfrac{1}{6}(e^9 - 1)}$$

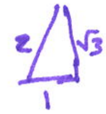
16. Find the area of the region inside $r = 1 + \cos\theta$ and outside $r = 3\cos\theta$. (5 points)



$1 + \cos\theta = 3\cos\theta$

$1 = 2\cos\theta$

$\frac{1}{2} = \cos\theta \quad \theta = \pi/3, -\pi/3$



$1 + \cos\theta = 0$

$\cos\theta = -1$

$\theta = \pi$

$$2\left[\int_{\pi/3}^{\pi/2}\int_{3\cos\theta}^{1+\cos\theta} r\,dr\,d\theta + \int_{\pi/2}^{\pi}\int_{0}^{1+\cos\theta} r\,dr\,d\theta\right] = \cancel{2}\left[\int_{\pi/3}^{\pi/2} \cancel{\tfrac{1}{2}}(1+\cos\theta)^2 - (3\cos\theta)^2\,d\theta + \int_{\pi/2}^{\pi} \cancel{\tfrac{1}{2}}(1+\cos\theta)^2\,d\theta\right]$$

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17. Rewrite the integral $\int_{-3}^{3}\int_{0}^{\sqrt{9-x^2}} \sin(x^2+y^2)\,dydx$ in polar coordinates. You do not need to evaluate. (4 points)

$$\int_0^{\pi}\int_0^{3} \sin(r^2)\,r\,dr\,d\theta$$

18. Evaluate $\int_1^2\int_0^{2z}\int_0^{\ln x} xe^{-y}\,dydxdz$. (5 points)

$$\int_1^2\int_0^{2z} -xe^{-y}\Big|_0^{\ln x} dxdz = \int_1^2\int_0^{2z} -xe^{-\ln x} + xe^{0}\Big| dxdz =$$

$$\int_1^2\int_0^{2z} -x\cdot\frac{1}{x} + x(1)\,dxdz = \int_1^2\int_0^{2z} x - 1\,dxdz = \int_1^2 \frac{1}{2}x^2 - x\Big|_0^{2z} dz$$

$$= \int_1^2 2z^2 - 2z\,dz = \frac{2}{3}z^3 - z^2\Big|_1^2 = \frac{16}{3} - 4 - \frac{2}{3} + 1 = \boxed{5/3}$$

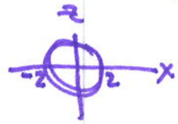
19. Set up the triple integral needed to find the volume enclosed by the surfaces $y = x^2 + z^2$ and $y = 8 - x^2 - z^2$. You do not need to evaluate it. (4 points)

$x^2 + z^2 = 8 - x^2 - z^2$

$2x^2 + 2z^2 = 8$

$x^2 + z^2 = 4$

$$\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{x^2+z^2}^{8-x^2-z^2} dydzdx$$



20. Set up the triple integral to find the volume of the region enclosed by $z = x^2 + y^2$ and $z = 36 - 3x^2 - 3y^2$ in cylindrical coordinates. You do not need to evaluate it. (4 points)

$36 - 3x^2 - 3y^2 = x^2 + y^2$

$36 = 4x^2 + 4y^2$

$9 = x^2 + y^2$

$z = r^2$

$z = 36 - 3r^2$

$$\int_0^{2\pi}\int_0^3\int_{r^2}^{36-3r^2} r\,dz\,dr\,d\theta$$

21. Convert the integral $\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{2-\sqrt{4-x^2-y^2}}^{2+\sqrt{4-x^2-y^2}}(x^2+y^2+z^2)^{\frac{3}{2}}\,dzdydx$ into spherical coordinates. You do not need to evaluate it. (4 points)

$z = 2 \pm \sqrt{4-x^2-y^2}$

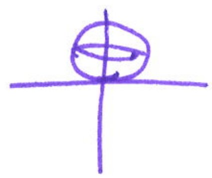
$(z-2) = \pm\sqrt{4-x^2-y^2}$

$x^2 + y^2 + (z-2)^2 = 4$

$x^2 + y^2 + z^2 - 4z + \cancel{4} = \cancel{4}$

$\rho^2 = 4\rho\cos\varphi$

$\rho = 4\cos\varphi$



$$\int_0^{2\pi}\int_0^{\pi/2}\int_0^{4\cos\varphi} \rho^3\,\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/2}\int_0^{4\cos\varphi} \rho^5\sin\varphi\,d\rho\,d\varphi\,d\theta$$

$(x^2+y^2+z^2)^{3/2} =$

$(\rho^2)^{3/2} = \rho^3$

16 cont'd

$$\int_{\pi/3}^{\pi/2} 1+2\cos\theta+\cos^2\theta-9\cos^2\theta \, d\theta + \int_{\pi/2}^{\pi} 1+2\cos\theta+\cos^2\theta \, d\theta$$

$$\int_{\pi/3}^{\pi/2} 1+2\cos\theta-8\cos^2\theta \, d\theta + \int_{\pi/2}^{\pi} 1+2\cos\theta+\cos^2\theta \, d\theta$$

$$\int_{\pi/3}^{\pi/2} 1+2\cos\theta-4-4\cos 2\theta \, d\theta + \int_{\pi/2}^{\pi} 1+2\cos\theta+\tfrac{1}{2}+\tfrac{1}{2}\cos 2\theta \, d\theta$$

$$-3\theta+2\sin\theta-2\sin 2\theta \Big|_{\pi/3}^{\pi/2} + \tfrac{3}{2}\theta + 2\sin\theta + \tfrac{1}{4}\sin 2\theta \Big|_{\pi/2}^{\pi}$$

$$-\cancel{\frac{3\pi}{2}} + \cancel{2} - 0 + \pi - \cancel{\sqrt{3}} + \cancel{\sqrt{3}} + \cancel{\frac{3\pi}{2}} + 0 + 0 - \frac{3\pi}{4} - \cancel{2} - 0 = \boxed{\frac{\pi}{4}}$$