Math 2568, Exam #2, Part I, Spring 2015 Name KEY

Instructions: Show all work. You may **not** use a calculator on this portion of the exam. **Give exact answers** (yes, that means fractions, square roots and exponentials, and not decimals). Reduce as much as possible. Be sure to complete all parts of each question. Provide explanations where requested. When you are finished with this portion of exam, get Part II.

1. Consider the following matrices: $A = \begin{bmatrix} 1 & -2 \\ 3 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 & 0 \\ 3 & -1 & 4 \end{bmatrix}, C = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & -2 \\ -1 & 1 & 1 \end{bmatrix}$, find each of the following matrices or say that they are undefined. If they are undefined, explain why. (5 points each)

   a. $C^T + 3I_3$

$$\begin{bmatrix} 1 & 3 & -1 \\ -1 & 0 & 1 \\ 2 & -2 & 1 \end{bmatrix} + \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 3 & -1 \\ -1 & 3 & 1 \\ 2 & -2 & 4 \end{bmatrix}$$

   b. AB

$$\underbrace{\underset{2\times 2}{\begin{bmatrix} 1 & -2 \\ 3 & 1 \end{bmatrix}}\underset{2\times 3}{\begin{bmatrix} 1 & 1 & 0 \\ 3 & -1 & 4 \end{bmatrix}}}_{\text{ok}} = \begin{bmatrix} 1-6 & 1+2 & 0-8 \\ 3+3 & 3-1 & 0+4 \end{bmatrix} = \begin{bmatrix} -5 & 3 & -8 \\ 6 & 2 & 4 \end{bmatrix}$$

   c. $B^T A$

$$\underbrace{\underset{3\times 2}{\begin{bmatrix} 1 & 3 \\ 1 & -1 \\ 0 & 4 \end{bmatrix}}\underset{2\times 2}{\begin{bmatrix} 1 & -2 \\ 3 & 1 \end{bmatrix}}}_{\text{ok}} = \begin{bmatrix} 1+9 & -2+3 \\ 1-3 & -2-1 \\ 0+12 & 0+4 \end{bmatrix} = \begin{bmatrix} 10 & 1 \\ -2 & -3 \\ 12 & 4 \end{bmatrix}$$

2. Give an example of two non-zero matrices whose produce is the zero matrix. (6 points)

$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ answers may vary

3. Find the inverse of the matrix $A = \begin{bmatrix} -3 & 1 \\ 4 & 1 \end{bmatrix}$. Use it to solve the system $\begin{bmatrix} -3 & 1 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 16 \end{bmatrix}$. (8 points)

$$A^{-1} = \frac{1}{-3-4}\begin{bmatrix} 1 & -1 \\ -4 & -3 \end{bmatrix} = \begin{bmatrix} -1/7 & 1/7 \\ 4/7 & 3/7 \end{bmatrix}\begin{bmatrix} 2 \\ 16 \end{bmatrix} =$$

$$\begin{bmatrix} -2/7 + 16/7 \\ 8/7 + 48/7 \end{bmatrix} = \begin{bmatrix} 14/7 \\ 56/7 \end{bmatrix} = \begin{bmatrix} 2 \\ 8 \end{bmatrix}$$

4. Determine if each statement is True or False. For each of the questions, assume that A is $n \times n$. (2 points each)

a. T F If A is invertible then A is row equivalent to I_n.

b. T F Two matrices, B which is $m \times n$ and C which is $p \times q$, produce a defined product when $m = q$.

c. T F If A has n pivots, then the system $A\vec{x} = \vec{0}$ does not have a unique solution.

d. T F If the columns of A span R^n then $Nul\ A = \{\vec{0}\}$.

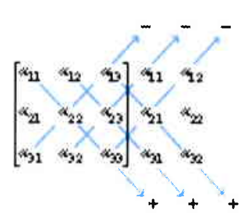
e. T F If A is onto, then A is one-to-one.

f. T F If there is a matrix C so that $CA = I$, then $Rank\ A = n$.

g. T F If the columns of A form a basis for R^n, then dim Nul A = n.

h. T F If A^T is invertible, then the columns of A^{-1} form a basis for R^n.

i. T F The method of finding the determinant of a 3x3 matrix shown in the attached image generalizes to any size matrix.



j. T F Row reducing a matrix does not change its determinant.

k. T F A system that does not have a unique solution cannot be solved with Cramer's rule.

l. T F A matrix is invertible if the determinant of the matrix is 0.

5. Find the determinant of the matrix $\begin{bmatrix}6&3&2&2&0\\5&0&-4&1&0\\4&-5&3&7&2\\3&0&0&1&0\\4&2&3&2&0\end{bmatrix}$ by the cofactor method. (10 points)

$$2\begin{vmatrix}6&3&2&2\\5&0&-4&1\\3&0&0&1\\4&2&3&2\end{vmatrix} = 2\left[3\begin{vmatrix}3&2&2\\0&-4&1\\2&3&2\end{vmatrix} - 1\begin{vmatrix}6&3&2\\5&0&-4\\4&2&3\end{vmatrix}\right] =$$

$$6\left[-4\begin{vmatrix}3&2\\2&2\end{vmatrix} - 1\begin{vmatrix}3&2\\2&3\end{vmatrix}\right] - 2\left[-5\begin{vmatrix}3&2\\2&3\end{vmatrix} + 4\begin{vmatrix}6&3\\4&2\end{vmatrix}\right] =$$

$$-24(6-4) - 6(9-4) + 10(9-4) - \cancel{8(12-12)} =$$

$$-24(2) - 6(5) + 10(5) = -48 - 30 + 50 = -48 + 20 = \boxed{-28}$$

6. Find the determinant of the matrix $\begin{bmatrix}1&-2&4\\3&1&-2\\2&0&1\end{bmatrix}$ by the row-reducing method. (10 points)

$-3R_1 + R_2 \to R_2$ no change $\quad -2R_1 + R_3 \to R_3$ no change

$\begin{bmatrix}1&-2&4\\0&7&-14\\0&4&-7\end{bmatrix}$ $\quad \frac{1}{7}R_2 \to R_2$ (multiply result by 7)

$\begin{bmatrix}1&-2&4\\0&1&-2\\0&4&-7\end{bmatrix}$ $\quad -4R_2 + R_2 \to R_3$ no change

$\begin{bmatrix}1&-2&4\\0&1&-2\\0&0&1\end{bmatrix}$ $\leftarrow$ det = 1 $\quad \times 7$

det of original matrix is $\boxed{7}$

7. Given that A and B are $n \times n$ matrices with det A = -3 and det B = 2, find the following. (4 points each)

a) det AB $= -6$

d) det B^T $= 2$

b) det A^{-1} $= -\frac{1}{3}$

e) det 2A $2^n(-3)$

c) det $(-AB^4)$

$(-1)^n(-3)\,2^4$

$= (-1)^n(-48)$

Math 2568, Exam #2, Part II, Spring 2015 Name ______________________

Instructions: Show all work. You **may** use a calculator on this portion of the exam. To show work on calculator problems, show the commands you used, and the resulting matrices. **Give exact answers** (yes, that means fractions, square roots and exponentials, and not decimals) unless specifically directed to give a decimal answer. This will require some operations to be done by hand even if not specifically directed to. Be sure to complete all parts of each question. Provide explanations where requested.

1. If a basis for R^3 is $B = \left\{ \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 4 \\ -3 \\ 0 \end{bmatrix} \right\}$, and given $[\vec{x}]_B = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$, find $\vec{x}$ in the standard basis.

   (6 points)

$$P_B = \begin{bmatrix} 1 & 3 & 4 \\ -1 & 0 & -3 \\ 3 & -2 & 0 \end{bmatrix}$$

$$P_B [\vec{x}]_B = \vec{x}$$

$$1\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} + 0\begin{bmatrix} 3 \\ 0 \\ -2 \end{bmatrix} + (-2)\begin{bmatrix} 4 \\ -3 \\ 0 \end{bmatrix} = \begin{bmatrix} 1+0-8 \\ -1+0+6 \\ 3+0+0 \end{bmatrix} = \begin{bmatrix} -7 \\ 5 \\ 3 \end{bmatrix}$$

2. If a vector in the standard basis is $\vec{x} = \begin{bmatrix} 2 \\ -4 \\ 5 \end{bmatrix}$, find its representation in the basis in problem #1.

   (7 points)

$$P_B^{-1} \vec{x} = [\vec{x}]_B$$

$$P_B^{-1} = \begin{bmatrix} 6/25 & 8/25 & 9/25 \\ 9/25 & 12/25 & 1/25 \\ -2/25 & -11/25 & -3/25 \end{bmatrix}$$

$$[\vec{x}]_B = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

3. Consider the basis $C = \left\{ \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix} \right\}$, and the vector $[\vec{x}]_C = \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix}$. Find the representation of the vector in the basis B in problem #1. (8 points)

$P_C[\vec{x}]_C = P_B[\vec{x}]_B$

$P_C = \begin{bmatrix} 1 & -1 & 4 \\ -1 & 2 & -1 \\ -2 & 3 & 2 \end{bmatrix}$

$P_B^{-1} P_C [\vec{x}]_C = [x]_B$

$P_B^{-1} P_C = \begin{bmatrix} -4/5 & 37/25 & 34/25 \\ -1/5 & 18/25 & 26/25 \\ 3/5 & -29/25 & -3/25 \end{bmatrix}$

$[\vec{x}]_B = \begin{bmatrix} 199/25 \\ 111/25 \\ -108/25 \end{bmatrix} = \begin{bmatrix} 7.96 \\ 4.44 \\ -4.32 \end{bmatrix}$

4. Use Cramer's rule to find the solution to the system $\begin{cases} x_1 + 3x_2 - 2x_3 + x_4 = 12 \\ 2x_1 - x_2 + x_3 + 4x_4 = 2 \\ 2x_2 - 3x_3 + 2x_4 = 12 \\ 3x_1 + 2x_3 - 5x_4 = -6 \end{cases}$. Write all the required matrices and their determinants, but you may calculate the determinants with your calculator. (15 points)

$A = \begin{bmatrix} 1 & 3 & -2 & 1 \\ 2 & -1 & 1 & 4 \\ 0 & 2 & -3 & 2 \\ 3 & 0 & 2 & -5 \end{bmatrix}$

$\det A = -88$

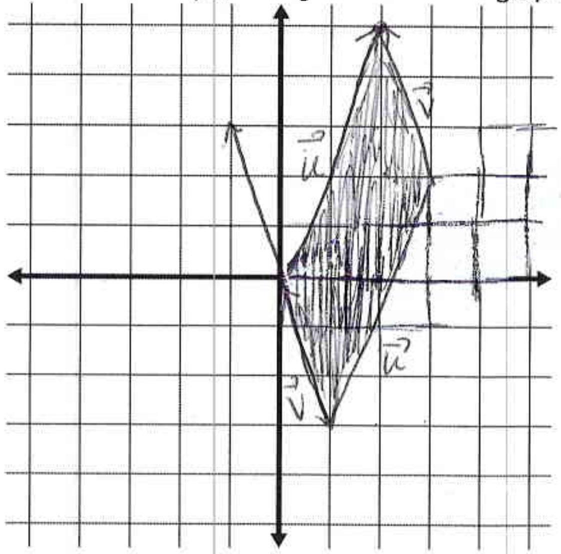
$x_1 = \dfrac{\begin{vmatrix} 12 & 3 & -2 & 1 \\ 2 & -1 & 1 & 4 \\ 12 & 2 & -3 & 2 \\ -6 & 0 & 2 & -5 \end{vmatrix}}{-88} = \dfrac{-88}{-88} = 1$

$x_2 = \dfrac{\begin{vmatrix} 1 & 12 & -2 & 1 \\ 2 & 2 & 1 & 4 \\ 0 & 12 & -3 & 2 \\ 3 & -6 & 2 & -5 \end{vmatrix}}{-88} = \dfrac{-176}{-88} = 2$

$x_3 = \dfrac{\begin{vmatrix} 1 & 3 & 12 & 1 \\ 2 & -1 & 2 & 4 \\ 0 & 2 & 12 & 2 \\ 3 & 0 & 6 & -5 \end{vmatrix}}{-88} = \dfrac{176}{-88} = -2$

$x_4 = \dfrac{\begin{vmatrix} 1 & 3 & -2 & 12 \\ 2 & -1 & 1 & 2 \\ 0 & 2 & -3 & 12 \\ 3 & 0 & 2 & -6 \end{vmatrix}}{-88} = \dfrac{-88}{-88} = 1$

5. Suppose that a parallelogram is bounded at one vertex by the vectors $\vec{u} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}, \vec{v} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$. Find the area of the parallelogram. Draw the graph below. (6 points)



$$\begin{bmatrix} 2 & -1 \\ 5 & 3 \end{bmatrix} \Rightarrow$$

$$6+5=11$$

6. A parallelepiped (slanted box) is defined in one corner by the vectors $\vec{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \vec{v} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \vec{w} = \begin{bmatrix} 3 \\ -2 \\ 5 \end{bmatrix}$.

(8 points)

$$\begin{vmatrix} 1 & -1 & 3 \\ 2 & 1 & -2 \\ 3 & 1 & 5 \end{vmatrix} = 1\begin{vmatrix} 1 & -2 \\ 1 & 5 \end{vmatrix} + 1\begin{vmatrix} 2 & -2 \\ 3 & 5 \end{vmatrix} + 3\begin{vmatrix} 2 & 1 \\ 3 & 1 \end{vmatrix} =$$

$$(5+2) + (10+6) + 3(2-3) =$$

$$7 + 16 + (-3) = 23 - 3 = 20$$