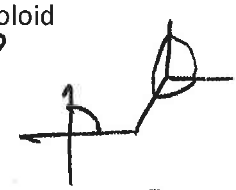
Math 2153, Final Exam, Spring 2014 Name KEY

Instructions: Show all work. Use exact answers unless specifically asked to round. You may check your answers in the calculator, but you must show work to get full credit. Incorrect answers with no work will receive no credit. Be sure to complete all the requested elements of each problem.

1. Find the center of mass for the region bounded by the coordinate planes, and the paraboloid $z = 9 - x^2 - y^2$. (22 points) assume density is constant $= \rho$



$z = 9 - r^2$

$$M = \iiint_Q \rho \, dV = \int_0^{\pi/2}\int_0^3\int_0^{9-r^2} \rho r \, dz \, dr \, d\theta =$$

$$= \rho \int_0^{\pi/2}\int_0^3 r(9-r^2)\, dr\, d\theta = \rho \int_0^{\pi/2}\int_0^3 9r - r^3 \, dr\, d\theta =$$

$$\rho \int_0^{\pi/2} \frac{9}{2}r^2 - \frac{r^4}{4}\Big|_0^3 d\theta = \frac{81\rho}{4}\int_0^{\pi/2} d\theta = \frac{81\pi\rho}{8}$$

$$M_{xy} = \int_0^{\pi/2}\int_0^3\int_0^{9-r^2} \rho r z \, dz\, dr\, d\theta = \rho \int_0^{\pi/2}\int_0^3 \frac{rz^2}{2}\Big|_0^{9-r^2} dr\, d\theta = \frac{\rho}{2}\int_0^{\pi/2}\int_0^3 81r - 18r^2 + r^3 \, dr\, d\theta$$

$$= \frac{\rho}{2}\int_0^{\pi/2} \frac{81r^2}{2} - 6r^3 + \frac{r^4}{4}\Big|_0^3 d\theta = \rho \int_0^{\pi/2} \frac{891}{8}\, d\theta = \frac{891\pi\rho}{16}$$

$$M_{yz} = \rho \int_0^{\pi/2}\int_0^3\int_0^{9-r^2} r^2\cos\theta \, dz\, dr\, d\theta = \rho \int_0^{\pi/2}\int_0^3 (9r^2 - r^4)\cos\theta \, dr\, d\theta =$$

$$= \rho \int_0^{\pi/2} \cos\theta\left(\frac{9}{3}r^3 - \frac{r^5}{5}\right)\Big|_0^3 d\theta = \rho \int_0^{\pi/2} \cos\theta \cdot \frac{162}{5}\, d\theta = \frac{162\rho}{5}\sin\theta\Big|_0^{\pi/2} = \frac{162\rho}{5}$$

$$M_{xz} = \rho \int_0^{\pi/2}\int_0^3\int_0^{9-r^2} r^2\sin\theta \, dz\, dr\, d\theta = \rho \int_0^{\pi/2}\int_0^3 (9r^2 - r^4)\sin\theta \, dr\, d\theta =$$

$$= \rho \int_0^{\pi/2} \sin\theta \left(3r^3 - \frac{r^5}{5}\right)\Big|_0^3 d\theta = \frac{162\rho}{5}\int_0^{\pi/2}\sin\theta\, d\theta = -\frac{162\rho}{5}(\cos\theta|_0^{\pi/2} = \frac{162\rho}{5}$$

$$\bar{x} = \frac{M_{yz}}{M} = \frac{\frac{162\rho}{5}}{\frac{81\pi\rho}{8}} = \frac{162\rho}{5}\cdot\frac{8}{81\pi\rho} = \frac{16}{5\pi} = \bar{y}$$

$$\frac{M_{xy}}{M} = \bar{z} = \frac{891\pi\rho}{16}\cdot\frac{8}{81\pi\rho} = \frac{11}{2}$$

$$\left(\frac{16}{5\pi}, \frac{16}{5\pi}, \frac{11}{2}\right)$$

2. Find the Jacobian for the change of variables $x = e^{2u+v}, y = e^{u+2v}$. (13 points)

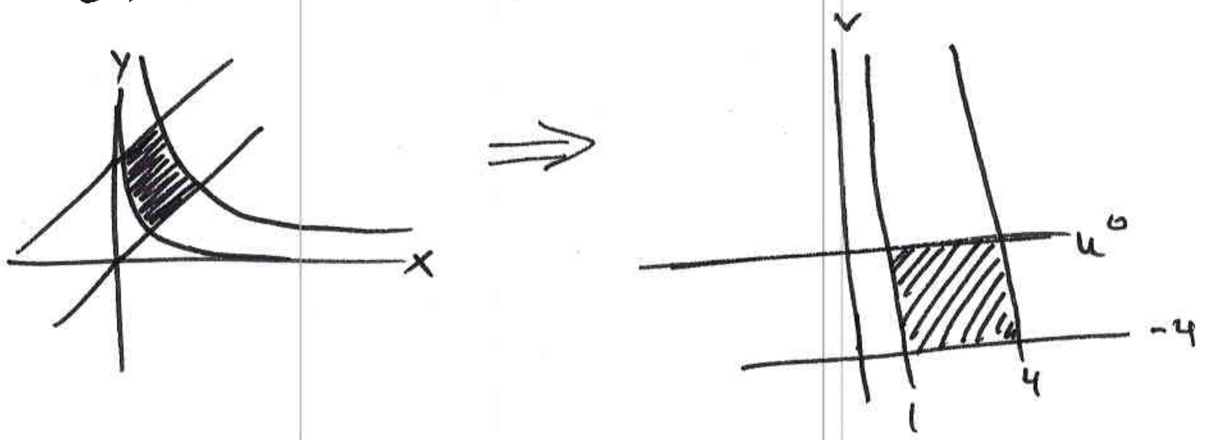
$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} e^{2u+v}\cdot 2 & e^{2u+v} \\ e^{u+2v} & e^{u+2v}\cdot 2 \end{vmatrix} = 4e^{3u+3v} - e^{3u+3v} = 3e^{3u+3v}$$

3. Find the change of coordinates needed for the region $xy = 1, xy = 4, y = x, y = x + 4$. Sketch the region, find x and y in terms of u and v, and the limits on u and v. (13 points)

$x - y = 0 \quad x - y = -4$

let $u = xy$, $v = x - y$

$[1, 4]$ $\qquad$ $[-4, 0]$



4. Minimize/Maximize the function $f(x, y) = x^2 y$ subject to the constraint $\frac{1}{x} + \frac{1}{y} = 1$. (16 points)

$\frac{1}{x} + \frac{1}{y} - 1 = 0$

$f_x = \lambda g_x \qquad 2xy = \frac{-\lambda}{x^2} \Rightarrow 2x^3 y = -\lambda \Rightarrow 2x^3 y = x^2 y^2$

$f_y = \lambda g_y \qquad x^2 = \frac{-\lambda}{y^2} \qquad x^2 y^2 = -\lambda$

$2x^3 y - x^2 y^2 = 0$

$x^2 y (2x - y) = 0 \rightarrow 2x - y = 0$

$\uparrow$ cannot be 0 in $\frac{1}{x} + \frac{1}{y} = 1$

$2x = y$

$\frac{1}{x} + \frac{1}{2x} = 1 \Rightarrow \frac{2}{2x} + \frac{1}{2x} = 1 \Rightarrow \frac{3}{2x} = 1$

$3 = 2x$

$\Rightarrow \frac{3}{2} = x$

$2x = y$

$2(\frac{3}{2}) = 3 = y$

$(3/2, 3, 27/4)$

5. Find all the critical points of the graph $f(x, y) = x^2y^2 - x$. Characterize each point as a maximum, a minimum or a saddle point (or cannot be determined) from the second partials test. (18 points)

$f_x = 2xy^2 - 1 = 0$

$f_y = 2x^2y = 0 \Rightarrow x=0$ or $y=0$

but if $x=0$ $-1=0$ is a contradiction

likewise if $y=0$ $-1=0$ is also a contraction

both equation are never zero at the same point $ so there is no min/max or saddle pt.

Since there are no critical points

6. Use the chain rule to find $\frac{df}{dt}$ for $f(x, y) = \sqrt{x^2 + y^2}, x = t, y = t^2$. (13 points)

$$\frac{\partial f}{\partial x} = \frac{1}{2}(x^2+y^2)^{-1/2} 2x = \frac{x}{\sqrt{x^2+y^2}} = \frac{t}{\sqrt{t^2+t^4}}$$

$$\frac{\partial f}{\partial y} = \frac{1}{2}(x^2+y^2)^{-1/2} \cdot 2y = \frac{y}{\sqrt{x^2+y^2}} = \frac{t^2}{\sqrt{t^2+t^4}}$$

$$\frac{dx}{dt} = 1 \qquad \frac{dy}{dt} = 2t$$

$$\frac{df}{dt} = \frac{1}{\sqrt{t^2+t^4}} + \frac{2t^3}{\sqrt{t^2+t^4}} = \frac{1+2t^3}{\sqrt{t^2+t^4}} = \frac{1+2t^3}{t\sqrt{1+t^2}}$$

7. Find the two implicit partial derivatives for the function $x\sqrt{z} + ze^y = x^4 + 1$. (15 points)

$F = x\sqrt{z} + ze^y - x^4 - 1$

$\frac{\partial F}{\partial x} = \sqrt{z} - 4x^3$

$\frac{\partial F}{\partial y} = ze^y$

$\frac{\partial F}{\partial z} = \frac{1}{2}\frac{x}{\sqrt{z}} + e^y$

$\frac{\partial z}{\partial x} = -\frac{\sqrt{z} - 4x^3}{\frac{x}{2\sqrt{z}} + e^y}$

$\frac{\partial z}{\partial y} = -\frac{ze^y}{\frac{x}{2\sqrt{z}} + e^y}$

8. Suppose a particle is launched with an acceleration vector $\vec{a}(t) = (t-1)\hat{\imath} + 2\hat{\jmath} - \hat{k}$. Find the position vector given the initial velocity vector is $\vec{v}(0) = 2\hat{\imath} + 3\hat{\jmath} - \hat{k}, \vec{r}(0) = -\hat{\jmath} + 20\hat{k}$. (20 points)

$\int (t-1)\hat{\imath} + 2\hat{\jmath} - \hat{k}\, dt = \left(\frac{t^2}{2} - t + C_1\right)\hat{\imath} + (2t + C_2)\hat{\jmath} + (-t + C_3)\hat{k}$

$C_1 = 2$ $C_2 = 3$ $C_3 = -1$

$v(t) = \left(\frac{1}{2}t^2 - t + 2\right)\hat{\imath} + (2t + 3)\hat{\jmath} + (-t - 1)\hat{k}$

$\int \left(\frac{1}{2}t^2 - t + 2\right)\hat{\imath} + (2t + 3)\hat{\jmath} + (-t - 1)\hat{k}\, dt =$

$\left(\frac{1}{6}t^3 - \frac{1}{2}t^2 + 2t + C_1\right)\hat{\imath} + (t^2 + 3t + C_2)\hat{\jmath} + \left(-\frac{1}{2}t^2 - t + C_3\right)\hat{k}$

$C_1 = 0$ $C_2 = -1$ $C_3 = 20$

$\vec{r}(t) = \left(\frac{1}{6}t^3 - \frac{1}{2}t^2 + 2t\right)\hat{\imath} + (t^2 + 3t - 1)\hat{\jmath} + \left(-\frac{1}{2}t^2 - t + 20\right)\hat{k}$

9. Find the symmetric form of the line perpendicular to the plane $2x - 4y + z = 12$ and passes through the point (-2,1,2). (13 points)

$\langle 2, -4, 1 \rangle$

$$\frac{x+2}{2} = \frac{y-1}{-4} = \frac{z-2}{1}$$

10. Find the limit of the following expressions at the indicated point. If the limit exists, state its value. If the limit does not exist, explain why not. You will need to test multiple paths. You may wish to use polar or spherical coordinates. (10 points each)

a. $\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^4 + y^2}$

$y = kx^2$ $\quad \lim_{x\to 0} \frac{x^2 \cdot kx^2}{x^4 + k^2x^4} = \lim_{x\to 0} \frac{kx^4}{x^4(1+k^2)}$

$= \frac{k}{1+k^2}$ which changes as k changes

DNE

b. $\lim_{(x,y)\to(0,0)} \frac{\sin\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}}$

let $r^2 = x^2 + y^2$

$\lim_{r\to 0} \frac{\sin r}{r} = 1$

Since the limit doesn't depend on how we get to origin, limit exists

11. Find both partial derivatives for $f(x, y) = xe^{\sin y}$. (12 points)

$\frac{\partial f}{\partial x} = e^{\sin y}$

$\frac{\partial f}{\partial y} = xe^{\sin y} \cdot \cos y$

12. Find the volume of the solid inside the cylinder $x^2 + y^2 = 1$ above the xy-plane and below the function $z = \frac{x}{\sqrt{x^2+y^2}}$. Set up a double or triple integral and compute the result. (18 points)

$x^2+y^2=1 \Rightarrow r=1$

$z = \frac{x\cos\theta}{x} = \cos\theta$

$$\int_0^{2\pi}\int_0^1\int_0^{\cos\theta} r\,dz\,dr\,d\theta =$$

$$\int_0^{2\pi}\int_0^1 r\cos\theta\,dr\,d\theta = \int_0^{2\pi}\frac{1}{2}\cos\theta\,d\theta = \frac{1}{2}\sin\theta\Big|_0^{2\pi} = 0$$

this isn't likely (do 1/4 of region & use symmetry)

$$4\int_0^{\pi/2}\int_0^1 r\cos\theta\,dr\,d\theta = 2\int_0^{\pi/2}\cos\theta\,d\theta = 2\sin\theta\Big|_0^{\pi/2} = 2$$

13. Integrate. $\int_0^{\pi}\int_0^{\pi}\int_0^{\sin\varphi}\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta$ (10 points)

$$\int_0^{\pi}\int_0^{\pi}\frac{\rho^3}{3}\sin\varphi\Big|_0^{\sin\varphi}d\varphi\,d\theta = \frac{1}{3}\int_0^{\pi}\int_0^{\pi}\sin^4\varphi\,d\varphi\,d\theta =$$

$\frac{1}{4}(1-\cos 2\varphi)^2$

$$\frac{1}{12}\int_0^{\pi}\int_0^{\pi}1-2\cos2\varphi+\cos^2 2\varphi\,d\varphi\,d\theta =$$

$\frac{1}{2}(1+\cos4\varphi)$

$$\frac{1}{12}\int_0^{\pi}\int_0^{\pi}\frac{3}{2}-2\cos2\varphi+\frac{1}{2}\cos4\varphi\,d\varphi\,d\theta = \frac{\pi}{12}\Big[\frac{3}{2}\varphi - \sin2\varphi + \frac{1}{8}\sin4\varphi\Big|_0^{\pi} =$$

$$= \boxed{\frac{\pi^2}{8}}$$

14. Determine if the vector field $\vec{G}(x,y) = 2xy\hat{\imath} + (x^2+1)\hat{\jmath} + 2z\hat{k}$ is conservative. If it is, find the potential function. (14 points)

$$\nabla\times G \quad \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2xy & x^2+1 & 2z \end{vmatrix} = (0-0)\hat{\imath} - (0-0)\hat{\jmath} + (2x-2x)\hat{k} = \vec{0}$$

conservative

$\int 2xy\,dx = x^2y$ + stuff w/ y & z

$\int x^2+1\,dy = x^2y + y$ + stuff w/ x & z

$\int 2z\,dz = z^2$ + stuff w/ x & y

$$g(x,y,z) = x^2y + y + z^2 + K$$

15. For the curve $\vec{r}(t) = sin^2 2t\hat{\imath} + cos^2 2t\hat{\jmath}$, find the unit tangent vector, and the unit normal vector. (16 points)

$$\vec{r}'(t) = 2\sin 2t \cos 2t \cdot 2\hat{\imath} + 2\cos 2t \sin 2t \cdot 2\hat{\jmath}$$

$$= 4\sin 2t \cos 2t (\hat{\imath} + \hat{\jmath})$$

$$\|\vec{r}'(t)\| = 4\sin 2t \cos 2t \sqrt{2}$$

$$\vec{T}(t) = \frac{1}{\sqrt{2}}\hat{\imath} + \frac{1}{\sqrt{2}}\hat{\jmath}$$

$$T'(t) = 0\hat{\imath} + 0\hat{\jmath}$$ ⇒ unit normal does not exis for this curve.

16. Find the equation of the tangent plane for the function $g(x,y) = ye^x - x^2$ at the point (0,3,3). Then find the equation of the line that is normal to the surface at the same point. (14 points)

$f_x = ye^x - 2x$

$f_y = e^x$

$G = ye^x - x^2 - z$

$\nabla G = \langle f_x, f_y, -1 \rangle$

$\nabla G(0,3,3) = \langle 3, 1, -1 \rangle$

$f_x(0,3) = 3$

$f_y(0,3) = 1$

tangent plane

$$3(x-0) + 1(y-3) - 1(z-3) = 0$$

normal line

$$\frac{x}{3} = \frac{y-3}{1} = \frac{z-3}{-1}$$ or

$$\vec{r}(t) = 3t\hat{\imath} + (t+3)\hat{\jmath} + (-t+3)\hat{k}$$

17. Find the value of $\int_C \vec{F} \cdot d\vec{r}$ for the vector field $\vec{F}(x,y,z) = \frac{1}{3}x^2z\hat{\imath} + \frac{1}{2}y\hat{\jmath} + \frac{1}{9}x^3\hat{k}$ from the point (1,-1,0) to the point (2,3,5). If the field is conservative, use the Fundamental Theorem of line integrals. If it is not, find the work done on the straight-line path between the two points. (15 points)

$$\nabla \times F = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \frac{1}{3}x^2z & \frac{1}{2}y & \frac{1}{9}x^3 \end{vmatrix} = (0-0)\hat{\imath} - (\tfrac{1}{3}x^2 - \tfrac{1}{3}x^2)\hat{\jmath} + (0-0)\hat{k}$$ conservative

$$\int \tfrac{1}{3}x^2 z\,dx = \tfrac{1}{9}x^3 z + \text{stuff} \qquad \int \tfrac{1}{2}y\,dy = \tfrac{1}{4}y^2 + \text{stuff}$$

$$\int \tfrac{1}{9}x^3\,dz = \tfrac{1}{9}x^3 z + \text{stuff} \qquad f(x,y,z) = \tfrac{1}{9}x^3 z + \tfrac{1}{4}y^2 + K$$

$$f(2,3,5) - f(1,-1,0) = \tfrac{1}{9}(2)^3(5) + \tfrac{1}{4}(3)^2 - \tfrac{1}{9}(1)^3(0) - \tfrac{1}{4}(-1)^2 =$$

$$\tfrac{40}{9} + \tfrac{9}{4} - \tfrac{1}{4} = \tfrac{58}{9}$$

18. Find the value of the line integral $\int_C (2y - 3xy)dx + (y + x)dy$ around the area bounded by the rectangle with vertices (0,0), (1,0), (1,7), (0,7), traversed from (0,0) counter-clockwise. (13 points)

$$\iint_R \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\, dA = \iint 1 - (2 - 3x)\, dA$$

$$1 - 2 + 3x$$

$$= \int_0^1 \int_0^7 -1 + 3x \, dy\, dx = \int_0^1 y(-1+3x)\Big|_0^7 =$$

$$7\int_0^1 -1 + 3x\, dx = 7\left[-x + \tfrac{3}{2}x^2\Big|_0^1\right] = 7\left[-1 + \tfrac{3}{2}\right] = \tfrac{7}{2}$$