Instructions: Show all work. Use exact answers unless specifically asked to round. You may check your answers in the calculator, but you must show work to get full credit. Incorrect answers with no work will receive no credit. Be sure to complete all the requested elements of each problem.

1. For the curve $\vec{r}(t) = \sin^3 t\,\hat{\imath} - \cos^3 t\,\hat{\jmath}$, find the unit tangent vector, the unit normal vector and the binormal vector. (14 points)

$$\vec{r}'(t) = 3\sin^2 t\cos t\,\hat{\imath} + 3\cos^2 t\sin t\,\hat{\jmath} = 3\sin t\cos t\,(\sin t\,\hat{\imath} + \cos t\,\hat{\jmath})$$

$$\|\vec{r}'(t)\| = 3\sin t\cos t\sqrt{\sin^2 t + \cos^2 t} = 3\sin t\cos t$$

$$\vec{T}(t) = \frac{3\sin^2 t\cos t\,\hat{\imath} + 3\cos^2 t\sin t\,\hat{\jmath}}{3\sin t\cos t} = \boxed{\sin t\,\hat{\imath} + \cos t\,\hat{\jmath}}$$

$$\vec{T}'(t) = \cos t\,\hat{\imath} - \sin t\,\hat{\jmath} \qquad \|T'(t)\| = 1$$

$$\vec{N}(t) = \boxed{\cos(t)\,\hat{\imath} - \sin t\,\hat{\jmath}}$$

$$\vec{B}(t) = \vec{T}(t) \times \vec{N}(t) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \sin t & \cos t & 0 \\ \cos t & -\sin t & 0 \end{vmatrix} = (-\sin^2 t + -\cos^2 t)\hat{k}$$

$$= \boxed{-1\hat{k}}$$

2. Find the directional derivative of the function $f(x,y) = x^2 e^{x+y}$ at the point $(1,-1,1)$ in the direction $\vec{u} = 2\hat{\imath} - 3\hat{\jmath}$. (8 points)

$$\nabla f = (2xe^{x+y} + x^2 e^{x+y})\hat{\imath} + (x^2 e^{x+y})\hat{\jmath}$$

$$\nabla f(1,-1) = (2+1)\hat{\imath} + (1)\hat{\jmath} = \langle 3, 1 \rangle$$

$$\hat{u} = \frac{2}{\sqrt{13}}\hat{\imath} - \frac{3}{\sqrt{13}}\hat{\jmath} = \left\langle \frac{2}{\sqrt{13}}, -\frac{3}{\sqrt{13}} \right\rangle$$

$$D_u f = \nabla f \cdot \hat{u} = \langle 3, 1 \rangle \cdot \left\langle \frac{2}{\sqrt{13}}, -\frac{3}{\sqrt{13}} \right\rangle = \frac{6}{\sqrt{13}} - \frac{3}{\sqrt{13}} = \frac{3}{\sqrt{13}}$$

3. Find the equation of the tangent plane for the function $g(x,y) = y\ln|x| - x^2$ at the point (1,6,-1). Then find the equation of the line that is normal to the surface at the same point. (12 points)

$F(x,y,z) = y\ln|x| - x^2 - z \qquad \nabla F = \langle \frac{y}{x} - 2x, \ln|x|, -1 \rangle$

$\nabla F(1,6,-1) = \langle 6-2, 0, -1 \rangle = \langle 4, 0, -1 \rangle$

tangent plane: $4(x-1) + 0(y-6) - 1(z+1) = 0$

or $\quad 4(x-1) - (z+1) = 0$

normal line $\qquad \vec{r}(t) = (4t+1)\hat{\imath} + (6)\hat{\jmath} + (-t-1)\hat{k}$

4. Find the equation of the tangent plane for the parametric surface $\vec{r}(u,v) = 2\cos u \sin v\,\hat{\imath} + 2\sin u \sin v\,\hat{\jmath} + 2\cos v\,\hat{k}$ at the point $(\sqrt{2}, 0, \sqrt{2})$. Then find the equation for the normal vector to the function at any point. (12 points)

$u = 0 \qquad 2(1)(\sqrt{\frac{1}{2}}) = \sqrt{2}$
$v = \pi/4 \qquad 2(0)(\sqrt{\frac{1}{2}}) = 0$
$2(\sqrt{\frac{1}{2}}) = \sqrt{2} \checkmark$

$\vec{r_u} = -2\sin u \sin v\,\hat{\imath} + 2\cos u \sin v\,\hat{\jmath} + 0\hat{k}$

$\vec{r_v} = 2\cos u \cos v\,\hat{\imath} + 2\sin u \cos v\,\hat{\jmath} - 2\sin v\,\hat{k}$

$\vec{r_u} \times \vec{r_v} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -2\sin u \sin v & 2\cos u \sin v & 0 \\ 2\cos u \cos v & 2\sin u \cos v & -2\sin v \end{vmatrix} = (-4\sin v \cos u)\hat{\imath} - (4\sin^2 v \sin u)\hat{\jmath} + \underbrace{(-4\sin^2 u \cos v \sin v - 4\cos^2 u \cos v \sin v)}_{=1}\hat{k}$

$\to \vec{N} = -4[\sin^2 v \cos u\,\hat{\imath} + \sin^2 v \sin u\,\hat{\jmath} + \cos v \sin v\,\hat{k}]$

$\vec{N} = -4\langle \frac{1}{2}(1), 0, \frac{1}{2} \rangle = \langle -2, 0, -2 \rangle$

tangent plane: $-2(x-\sqrt{2}) + 0(y-0) - 2(z - \sqrt{2}) = 0 \quad$ or

$-2(x-\sqrt{2}) - 2(z - \sqrt{2}) = 0$

5. Find the length of the parametric curve $\vec{r}(t) = t\hat{i} + \ln|\cos t|\hat{j}$. (8 points)

$$\vec{r}'(t) = 1\hat{i} - \tan t\,\hat{j}$$

$$\|\vec{r}'(t)\| = \sqrt{1 + \tan^2 t} = \sqrt{\sec^2 t} = |\sec t|$$

$$S = \int_a^b |\sec t|\,dt \qquad \text{for } a, b \text{ between } -\tfrac{\pi}{2} < t < \tfrac{\pi}{2}$$

6. Find the curvature of the function $\vec{r}(t) = t\sin t\,\hat{i} - \cos t\,\hat{j} + t^2\hat{k}$. Use the formula $\kappa = \frac{\|\vec{r}' \times \vec{r}''\|}{\|\vec{r}'\|^3}$.
Note any values of the parameter where the curvature is zero. (12 points)

$$\vec{r}'(t) = (\sin t + t\cos t)\hat{i} + \sin t\,\hat{j} + 2t\hat{k}$$

$$r''(t) = (2\cos t - t\sin t)\hat{i} + \cos t\,\hat{j} + 2\hat{k}$$

$$\|r'(t)\| = \sqrt{(\sin t + t\cos t)^2 + \sin^2 t + 4t^2} = \sqrt{\sin^2 t + 2t\cos t \sin t + t^2\cos^2 t + \sin^2 t + 4t^2}$$

$$= \sqrt{2\sin^2 t + 2t\cos t \sin t + t^2\cos^2 t + 4t^2}$$

$$r' \times r'' = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \sin t + t\cos t & \sin t & 2t \\ 2\cos t - t\sin t & \cos t & 2 \end{vmatrix}$$

$= (2\sin t - 2t\cos t)\hat{i} - (2\sin t + 2t\cos t - 4t\cos t + 2t^2\sin t)\hat{j}$
$\quad + (\sin t\cos t + t\cos^2 t - 2\sin t\cos t + t\sin^2 t)\hat{k}$

$= (2\sin t - 2t\cos t)\hat{i} + (2t\cos t - 2\sin t - 2t^2\sin t)\hat{j}$
$\quad + (t - \sin t\cos t)\hat{k}$

$$\|r' \times r''\| = \sqrt{(2\sin t - 2t\cos t)^2 + (2t\cos t - 2\sin t - 2t^2\sin t)^2 + (t - \sin t\cos t)^2} \qquad \text{Cont'd on extra page}$$

7. Find the surface area of the surface $x^2 + y^2 - z^2 = 1$ above the xy-plane and below the line z=4. (10 points)

$$x^2 + y^2 = 1 + z^2 \overset{z=4}{\Rightarrow} 17$$
$$x^2 + y^2 = 1 + z^2 \Rightarrow 1$$
$$z=0$$

$$\sqrt{x^2 + y^2 - 1} = \sqrt{z^2} \qquad z = \sqrt{x^2 + y^2 - 1}$$

$$\nabla F = \left\langle \frac{x}{\sqrt{x^2+y^2-1}}, \frac{y}{\sqrt{x^2+y^2-1}}, -1 \right\rangle$$

$$\iint_S \sqrt{1 + \frac{x^2}{x^2+y^2-1} + \frac{y^2}{x^2+y^2-1}}\,dA = \iint_S \sqrt{\frac{x^2+y^2-1 + x^2+y^2}{x^2+y^2-1}}\,dA = \int_0^{2\pi}\int_1^{\sqrt{17}} \sqrt{\frac{2r^2-1}{r^2-1}}\,r\,dr\,d\theta$$

you may use calculator at this point (watch out for parentheses

$\approx 2\pi\,(205.192277) \approx 1289.26$

6 continued

$$\|r' \times r''\| = \sqrt{\begin{array}{l}4\sin^2 t - 8t\sin t\cos t + 4t^2\cos^2 t + 4t^2\cos^2 t + 4\sin^2 t + 4t^4\sin^2 t - 8t\sin t\cos t \\ -8t^3\sin t\cos t + 8t^2\sin^2 t + t^2 - 2t\sin t\cos t + \cos^2 t\sin^2 t\end{array}}$$

$$= \sqrt{\begin{array}{l}8\sin^2 t - 18t\sin t\cos t + 8t^2\cos^2 t + 4t^4\sin^2 t - 8t^3\sin t\cos t + \underbrace{8t^2\sin^2 t}_{= 8t^2} \\ + t^2 + \cos^2 t\sin^2 t\end{array}}$$

$$= \sqrt{\sin^2 t\,(8+4t^4)\; -\sin t\cos t\,(18t + 8t^3) + \cos^2 t\sin^2 t \quad +9t^2}$$

$$K = \frac{\sqrt{(8+4t^4)\sin^2 t - \sin t\cos t\,(18t + 8t^3) + \cos^2 t\sin^2 t + 9t^2}}{\left(2\sin^2 t + 2t\cos t\sin t + t^2\cos^2 t + 4t^2\right)^{3/2}}$$

to find any points where the curvature is zero, graph any
version of the numerator (w/ or w/o the square root)

only obvious value is at $t=0$ (since at t terms are 0 $\$$ $\sin(0)=0$)

and there are no others.

8. Find the value of $\int_C \vec{F} \cdot d\vec{r}$ for the vector field $\vec{F}(x,y,z) = \frac{1}{3}x^2 z\hat{\imath} + \frac{1}{2}y\hat{\jmath} + \frac{1}{9}x^3\hat{k}$ from the point $(2,-1,0)$ to the point $(1,1,3)$. If the field is conservative, use the Fundamental Theorem of line integrals. If it is not, find the work done on the straight-line path between the two points. (12 points)

$\int \frac{1}{3}x^2 z \, dx = \frac{1}{9}x^3 z + g(y,z)$ $\quad \int \frac{1}{2}y \, dy = \frac{1}{4}y^2 + h(x,z)$

$\int \frac{1}{9}x^3 \, dz = \frac{1}{9}x^3 z + j(x,y)$

$f(x,y,z) = \frac{1}{9}x^3 z + \frac{1}{4}y^2$

by fundamental theorem of line integrals

potential function exists
field is conservative

$\int_C \vec{F} \cdot d\vec{r} = f(1,1,3) - f(2,-1,0) = \frac{1}{9}(1)^3(3) + \frac{1}{4}(1)^2 - \frac{1}{9}(2)^3(0) - \frac{1}{4}(-1)^2$

$= \frac{1}{3} + \frac{1}{4} - 0 - \frac{1}{4} = \boxed{\frac{1}{3}}$

9. Find the value of the line integral $\int_C (2x - y)dx + (y + 5x)dy$ around the area bounded by the rectangle with vertices $(0,0)$, $(2,0)$, $(2,5)$, $(0,5)$, traversed from $(0,0)$ counter-clockwise. (10 points)

$\frac{\partial N}{\partial x} = 5 \qquad \frac{\partial M}{\partial y} = -1 \qquad \iint_R \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \, dA = \int_0^2 \int_0^5 5 - (-1) \, dy \, dx =$

$6 \int_0^2 \int_0^5 dy \, dx = 6(2)(5) = 60$

10. Compute the value of the surface integral $\iint_S g(x,y,z)dS$ for $g = xy$ over the triangle $x + y + z = 1, x, y, z \geq 0$. (10 points)

$x + y = 1 \qquad \nabla F = \langle 1, 1, 1 \rangle$

$y = -x + 1 \qquad \|\nabla F\| = \sqrt{3}$

$\int_0^1 \int_0^{-x+1} \sqrt{3} \, xy \, dy \, dx = \frac{\sqrt{3}}{2} \int_0^1 xy^2 \Big|_0^{-x+1} dx = \frac{\sqrt{3}}{2} \int_0^1 x^3 - 2x^2 + x \, dx$

$\frac{\sqrt{3}}{2} \left[\frac{1}{4}x^4 - \frac{2}{3}x^3 + \frac{1}{2}x^2 \right]_0^1 = \frac{\sqrt{3}}{2} \left[\frac{1}{4} - \frac{2}{3} + \frac{1}{2} \right] = \frac{\sqrt{3}}{2} \left[\frac{1}{12} \right] = \frac{\sqrt{3}}{24}$

11. Find the value of the surface integral $\int\int_S \|\vec{r_u} \times \vec{r_v}\| dS$ for $\vec{r}(u,v) = uv\hat{i} + (u+v)\hat{j} + (u-v)\hat{k}$, for $0 \le u \le 3, 1 \le v \le 5$. (12 points)

$$\vec{r_u} = v\hat{i} + \hat{j} + \hat{k} \qquad \vec{r_v} = u\hat{i} + \hat{j} - \hat{k}$$

$$\vec{r_u} \times \vec{r_v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ v & 1 & 1 \\ u & 1 & -1 \end{vmatrix} = (-1-1)\hat{i} - (-v-u)\hat{j} + (v-u)\hat{k} =$$

$$(-2)\hat{i} + (u+v)\hat{j} + (v-u)\hat{k}$$

$$\|\vec{r_u} \times \vec{r_v}\| = \sqrt{4 + u^2 + v^2 + 2uv + u^2 + v^2 - 2uv} = \sqrt{4 + 2u^2 + 2v^2}$$

$$dS = \|\vec{r_u} \times \vec{r_v}\| dA \implies \int\int_S \|\vec{r_u} \times \vec{r_v}\| dS = \int\int_S (4 + 2u^2 + 2v^2) \, du\, dv$$

$$\int_0^3 \int_1^5 4 + 2u^2 + 2v^2 \, dv\, du = \int_0^3 16 + 8u^2 + \frac{248}{3} \, du = \frac{296}{3}(3) + 72$$
$$\underset{\frac{2}{3}v^3}{} \qquad \underset{\frac{8}{3}u^3}{} \qquad = 368$$

12. Find the flux $\int\int_S \vec{F} \cdot \vec{N} dS$ by the Divergence Theorem for $\vec{F}(x,y,z) = x^2\hat{i} + y^2\hat{j} + z^2\hat{k}$ on the unit sphere. (10 points)

$$\text{div } F = 2x + 2y + 2z \qquad\qquad \int\int_S \vec{F} \cdot \vec{N} dS = \int\int\int_Q \text{div} F \, dV$$

$$\int\int\int_Q 2x + 2y + 2z \, dV = \int_0^\pi \int_0^{2\pi} \int_0^1 (2\rho\cos\theta\sin\varphi + 2\rho\sin\theta\sin\varphi + 2\rho\cos\varphi)\rho^2 \sin\varphi \, d\rho\, d\theta\, d\varphi$$

$$2\int_0^\pi \int_0^{2\pi} \int_0^1 \rho^3 (\cos\theta\sin^2\varphi + \sin\theta\sin^2\varphi + \cos\varphi\sin\varphi) \, d\rho\, d\theta\, d\varphi =$$
$$\underset{\frac{1}{4}\rho^4|_0^1}{}$$

$$\frac{1}{2}\int_0^\pi \int_0^{2\pi} \cos\theta\sin^2\varphi + \sin\theta\sin^2\varphi + \cos\varphi\sin\varphi \, d\theta\, d\varphi =$$

$$\frac{1}{2}\int_0^\pi \left[\sin\theta\sin^2\varphi + -\cos\theta\sin^2\varphi \right]_0^{2\pi} + 2\pi\cos\varphi\sin\varphi \, d\varphi =$$
$$-(1-1)$$

$$\frac{2\pi}{2}\int_0^\pi \cos\varphi\sin\varphi \, d\varphi = \pi \left[\frac{1}{2}\sin^2\varphi \right]_0^\pi = 0$$

13. Use Stokes' Theorem to compute $\int_C \vec{F} \cdot d\vec{r} = \int \int_S (\nabla \times \vec{F}) \cdot \vec{N} dS$ for $\vec{F}(x,y,z) = x^2\hat{\imath} + y^2\hat{\jmath} + z^2\hat{k}$ over the top half of the unit sphere. (13 points)

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2 & y^2 & z^2 \end{vmatrix} = 0\hat{\imath} - 0\hat{\jmath} - 0\hat{k}$$

$$\int\int_S (\vec{\nabla} \times \vec{F}) \cdot \vec{N} dS = \int\int_S 0 \cdot \vec{N} dS = 0$$

(This makes sense since F is conservative on a closed curved $\Rightarrow$ it should be zero).